Note: $a < b$ is equivalently expressed as $b > a$ (b is greater than a).

The law of trichotomy states that a real number a is one of the following: $a < 0$, $a = 0$, $0 < a$, i.e., $a < 0$, $a = 0$, $a > 0$

a is said to be positive if $a > 0$

a is said to be negative if $a < 0$

We use the symbol $a \geq 0$ to mean that a is either positive or zero, $a \leq 0$ to mean a is either negative or zero.

If $a, b, c \in \mathbb{R}$ and $a < c$, $c < b$ both hold, we write $a < c < b$ and say c lies between a and b.

Real valued functions defined on an interval, Limit of a function (Cauchy's definition) and Algebra of limits

Here we will always write the set of all real numbers as $\mathbb{R}$.

Let $a, b \in \mathbb{R}$, we define (a, b), called the open interval (a, b) which is $(a, b) = \{x \in \mathbb{R} : a < x < b\}$. We define $[a, b]$, called the closed interval $[a, b]$ as $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Also we define two interval $[a, b)$ and $(a, b]$, called semi-open intervals and they are defined by

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\} \text{ and } (a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

Also, we define $(-\infty, a)$ and $(-\infty, a]$ by

$$(-\infty, a) = \{x \in \mathbb{R} : x < a\} \text{ and } (-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$$

Similarly, we define $(a, \infty) = \{x \in \mathbb{R} : x > a\}$ and $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$

$\mathbb{R}$ is written as $(-\infty, \infty)$. The sets mentioned above are called intervals.

Example: Let $I = (2, 3)$. Then $I = \{x \in \mathbb{R} : 2 < x < 3\}$

Let $I = (2, \infty)$. Then $I = \{x \in \mathbb{R} : x > 2\}$

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Let $I = [1, 3]$. Then $I = \{x \in \mathbb{R} : 1 \le x \le 3\}$

Here we will consider real valued functions defined on an interval.

Examples: 1. $f : [1,2] \to \mathbb{R}$ defined by

$$f(x) = x^2 \qquad 1 \le x \le 2$$

2. $f : (1, 2) \to \mathbb{R}$ defined by

$$f(x) = 2x+1, \quad 1 < x < 2$$

2. $f : (-\infty, 5) \to \mathbb{R}$ defined by

$$f(x) = x^2 + 2x + 1, \quad x < 5$$

3. $f : (3, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{x^2 - 9}{x - 3}, \quad x > 3$$

4. $f : (-\infty, 5] \to \mathbb{R}$ defined by

$$f(x) = x^2 - 5x, \quad x \le 5$$

5. $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \frac{x}{2}, & x > 2 \end{cases}$$

Cauchy's definition of limit of a function:

Let $f : [a, b] \to \mathbb{R}$ be a real valued function defined on the closed interval $[a, b]$, $a, b \in \mathbb{R}$. Let $c \in [a, b]$ such that $a < c < b$. A real number l is said to be a limit of f at c, written as

~~for~~ $\lim_{x \to c} f(x) = l$, if (we sometimes also ~~that~~ say that $f(x)$ tend to a limit l as x approaches c)

if given a positive number ϵ (epsilon), there exists a positive number δ (delta) such that

$$|f(x) - l| < \epsilon \quad \text{for } x \text{ in } 0 < |x-c| < \delta$$

i.e., if given a positive number ϵ, there exists a positive number δ such that $l-\epsilon < f(x) < l+\epsilon$ for x such that $c-\delta < x < c$ or $c < x < c+\delta$ (or for $x \in (c-\delta, c+\delta)$ but $x \neq c$)

[Note: Let $a \in \mathbb{R}$, The absolute value of a or modulus of a, denoted by $|a|$ and pronounced as mod a, is defined by

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

It has the properties : (i) $|-a| = |a|$ for all $a \in \mathbb{R}$

(ii) $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$

(iii) if $a, c \in \mathbb{R}$ and $c > 0$, then $|a| < c \Leftrightarrow -c < a < c$

($\Leftrightarrow$ means implies and implied by)

(iv) $-|a| \leq a \leq |a|$ for all $a \in \mathbb{R}$

So, $|f(x) - l| < \epsilon \Rightarrow -\epsilon < f(x) - l < \epsilon$

or, $l - \epsilon < f(x) < l + \epsilon$

$0 < |x-c| \Rightarrow x \neq c$ and $|x-c| < \delta$

$\Rightarrow c-\delta < x < c+\delta$

So, $0 < |x-c| < \delta \Rightarrow x \in (c-\delta, c+\delta)$ and $x \neq 0$

i.e., $c-\delta < x < c$ or $c < x < c+\delta$.]

$f(x)$ is said to have right hand limit l as x tends to c if given a positive number ϵ, there exists a positive number δ such that

$$|f(x) - l| < \epsilon \quad \text{for } x \text{ in } c < x < c+\delta$$

and it written as $\lim\limits_{x \to c+} f(x) = l$

Similarly, $f(x)$ is said to have a left hand limit l if given a positive number ϵ there exists a positive number δ such that

$$|f(x)-l|<\epsilon \quad \text{for } x \text{ in } c-\delta<x<c$$

and it written as $\lim\limits_{x\to c-} f(x) = l$

So, $f(x)$ has a limit l as x tends to c if and only if $f(x)$ has a right hand limit and a left hand limit as x tends to c and they are equal to l.

In terms of notation,

$$\lim_{x\to c} f(x) = l \text{ if and only if } \lim_{x\to c-} f(x) = \lim_{x\to c+} f(x) = l$$

If $c=a$, then $f(x)$ has a limit l as x tends to a if $\lim\limits_{x\to a+} f(x) = l$ i.e., given any positive ϵ, $\exists$ a positive δ such that

$$|f(x)-l|<\epsilon \quad \text{for } x \text{ in } a<x<a+\delta$$

Here the function is not defined when $x<a$

Similarly If $c=b$, then $f(x)$ has a limit l as x tends to b if $\lim\limits_{x\to b-} f(x) = l$, i.e., Given an $\epsilon>0$, there exist a $\delta>0$ such that

$$|f(x)-l|<\epsilon \quad \text{for } x \text{ in } b-\delta<x<b$$

Here the function is not defined when $x>b$

Worked Examples 1. $\lim\limits_{x\to 2} f(x) = 4$, where $f(x) = \frac{x^2-4}{x-2}$, $x\neq 2$
$= 10$, $x=2$

Here the domain of f is $\mathbb{R}$

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