Solution: when $x \neq 2$, $|f(x)-4| = \left|\frac{x^2-4}{x-2} - 4\right| = |x+2-4| = |x-2|$.

Let us choose $\epsilon > 0$.

$|f(x)-4| < \epsilon$ whenever $|x-2| < \epsilon$ and $x \neq 2$. Taking $\delta = \epsilon$. Then $\delta > 0$ and $|f(x)-4| < \epsilon$ for $0 < |x-2| < \delta$. So, $\lim_{x \to 2} f(x) = 4$

2. Show that $\lim_{x \to 0} f(x) = 0$ where $f(x) = \sqrt{x}$, $x \geq 0$

Solution:

Let us choose $\epsilon > 0$.

when $x \geq 0$, $|f(x) - 0| = \sqrt{x}$

Therefore, $|f(x) - 0| < \epsilon$ for all x satisfying $0 < x < \epsilon^2$, i.e., taking $\delta = \epsilon^2$, $|f(x) - 0| < \epsilon$ for x satisfying $0 < x < \epsilon^2 = \delta$

So, we have $\lim_{x \to 0} f(x) = 0$ as $f(x)$ is defined for $x \geq 0$.

Note: If $f : [a, b] \to \mathbb{R}$ be a function and then

$$\lim_{x \to a+} f(x) = \lim_{x \to a} f(x) \text{ and } \lim_{x \to b-} f(x) = \lim_{x \to b} f(x)$$

Algebra of limits

Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be two real valued functions defined on an interval I. Let $c \in I$ and

$\lim_{x \to c} f(x) = l_1$ and $\lim_{x \to c} g(x) = l_2$

Then (i) $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = l_1 + l_2$

(ii) $\lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = l_1 - l_2$

(iii) $\lim_{x \to c} (f(x) \cdot g(x)) = \left(\lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)\right) = l_1 l_2$

(iv) $\lim_{x \to c} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{l_1}{l_2}$ provided $l_2 \neq 0$

Continuity at a point

Let $f : [a,b] \to \mathbb{R}$ be a real valued function. Let $c \in [a,b]$ such that $a < c < b$. Then f is said to be continuous at c if $\lim_{x \to c} f(x) = f(c)$,

i.e., Given an $\epsilon > 0$, there exists a positive number $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for x in $|x - c| < \delta$.

f is said to be continuous at a if

$\lim_{x \to a+} f(x) = f(a)$, i.e Given an $\epsilon > 0$, there exists a positive number $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ for } x \text{ in } a < x < a+\delta.$$

f is said to be continuous at b if

$\lim_{x \to b-} f(x) = f(b)$, i.e., given an $\epsilon > 0$ There exists positive number δ such that

$$|f(x) - f(b)| < \epsilon \text{ for } x \text{ in } b - \delta < x < b$$

Continuity in an interval

Let $f : [a,b] \to \mathbb{R}$ be a real valued function. Then f is said to be continuous in $[a,b]$ if it is continuous at all points of $[a,b]$. So, f is continuous in $[a,b]$ if and only if

$$\lim_{x \to c} f(x) = f(c) \text{ for } a < c < b$$

and $\lim_{x \to a+} f(x) = f(a)$

and $\lim_{x \to b-} f(x) = f(b)$.

Worked examples

1. Discuss the continuity of $f(x) = |x|$ at $x = 0$

Solution: Choose an $\epsilon > 0$.

Now $|f(x) - f(0)| < \epsilon$

i.e., if $||x| - 0| < \epsilon$

i.e., if $|x - 0| < \epsilon$

So, if we take $\delta = \epsilon$, then $\delta > 0$

and $|f(x) - f(0)| < \epsilon$ for x in $|x - 0| < \delta$

So, $\lim_{x \to 0} f(x) = f(0)$, So, f is continuous at $x = 0$.

Note: Let $f : [a, b] \to \mathbb{R}$ be a real valued function and let $c \in [a, b]$ such that $a < c < b$. Then f is said to be right continuous at c if $\lim_{x \to c+} f(x) = f(c)$ and is said to be left continuous at c if $\lim_{x \to c-} f(x) = f(c)$

2. Show that $f(x) = \begin{cases} x^2 & \text{when } 0 < x < 1 \\ x & \text{when } 1 \le x < 2 \\ \frac{1}{4}x^2 & \text{when } 2 \le x < 3 \end{cases}$

is continuous at $x = 1$ but not continuous at $x = 2$

Solution: $f(1) = 1$, $\lim_{x \to 1-} f(x) = \lim_{x \to 1-} x^2 = 1$

$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} x = 1$. So, $\lim_{x \to 1} f(x) = 1 = f(1)$

So, f is continuous at $x = 1$

$f(2) = \frac{1}{4} 2^2 = 1$ $\quad \lim_{x \to 2-} f(x) = \lim_{x \to 2-} x = 2$

$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} \frac{1}{4} x^2 = 1$ $\quad$ So, $\lim_{x \to 2-} f(x) \neq \lim_{x \to 2+} f(x)$

So, f is not continuous at $x = 2$

Acquaintance with ~~basic~~ important properties of continuous on closed intervals.

Theorem 1 Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be two real valued functions which are continuous at $c\in[a,b]$

Then (i) $f(x)\pm g(x)$ is continuous at $x=c$

(ii) $f(x)\cdot g(x)$ is continuous at $x=c$

(iii) $f(x)/g(x)$ is continuous at $x=c$ if $g(x)\neq 0$ for $x\in[a,b]$

Theorem 2 Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be two real valued continuous function on the interval $[a,b]$, then

(i) $f(x)\pm g(x)$ is continuous on $[a,b]$

(ii) $f(x)\cdot g(x)$ is continuous on $[a,b]$

(iii) $f(x)/g(x)$ is continuous on $[a,b]$ if $g(x)\neq 0$ for $x\in[a,b]$.

Theorem 3 If $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ then f is bounded on $[a,b]$.

Theorem 4 If f is $[a,b]\to\mathbb{R}$ be continuous at $c\in[a,b]$ such that $a<c<b$ and $f(c)\neq 0$, then $f(x)$ will keep the same sign as $f(c)$ in a suitably restricted interval about c.

Theorem 5 Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Suppose $f(a)$ and $f(b)$ are of opposite signs. Then there exists at least one point c such that $a<c<b$ such that $f(c)=0$

Theorem 6 (Intermediate value Theorem) If $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and $f(a)\neq f(b)$. Then as x goes from a to b, $f(x)$ assumes at least once every value between $f(a)$ and $f(b)$