Theorem 7 Let $I$ be an interval and $f: I \to \mathbb{R}$ be continuous. Then $f(I) = \{ f(x) : x \in I \}$ is an interval.

Theorem 8 Let $I = [a, b]$ be a closed and bounded interval and a function $f: I \to \mathbb{R}$ be continuous. Then $f$ is bounded on $I$ and assume its greatest lower bound and least upper bound

[Note: Let $f: A \to \mathbb{R}$ be a real valued function ($D \subset \mathbb{R}$) then $f$ is said to be bounded if there exist $m$ and $M$ such that $m \le f(x) \le M$ for all $x \in A$. $m$ is called a lower bound and $M$ is called an upper bound. A lower bound $m$ is called a greatest lower bound if it is a lower bound, i.e., $m \le f(x)$ for all $x \in A$ and $p$ be any other lower bound then $p \le m$. Similarly, we define an upper bound $M$ a ~~greatest~~ least upper bound if $M$ is an upper bound, i.e., $f(x) \le M$ for all $x \in A$ and if $L$ be an any other upper bound then $M \le L$. Greatest lower bound and least upper bound are also called ~~supremum or infimum~~ infimum and supremum]

Theorem 8 Let $[a, b]$ be a closed and bounded interval and a function $f: [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $M$ and $m$ be the supremum and infimum of $f(x)$ and if $M \ne m$ then if $\mu$ be a number such that $m < \mu < M$ then there exists a point $p$ such that $a < p < b$ and $f(p) = \mu$

Discontinuity and types of discontinuity

A function is said to be discontinuous at a point if it is not continuous at that point.

There are three types of discontinuities :

1. Removable discontinuity
2. Discontinuity of the first kind
3. Discontinuity of the second kind.

1. Removable discontinuity

A function $f(x)$ has a removable discontinuity at $x=c$ if $\lim\limits_{x\to c} f(x) \neq f(c)$

This kind of discontinuity can be removed by changing the value of the function at $x=c$

For example, if $f(x) = \begin{cases} 2, & x=3 \\ x^2, & x\neq 3 \end{cases}$

Here $\lim\limits_{x\to 3} f(x) = 9 \neq f(3) = 2$

So, if we put $f(x) = \begin{cases} 9, & x=3 \\ x^2, & x\neq 3 \end{cases}$, the discontinuity at $x=3$ is removed.

So, for $f(x)$, $x=3$ is a removable discontinuity.

2. Discontinuity of the first kind

$f(x)$ is said to have discontinuity of first kind at $x=c$ if $\lim\limits_{x\to c+} f(x) \neq \lim\limits_{x\to c-} f(x)$

For example, let $f(x) = \begin{cases} x^2, & x\leq 1 \\ 5-x; & x>1 \end{cases}$

Then $\lim\limits_{x\to 1-} f(x) = 1$ and $\lim\limits_{x\to 1+} f(x) = 5-1 = 4$

So, $\lim\limits_{x\to 1-} f(x) \neq \lim\limits_{x\to 1+} f(x)$. So, $f(x)$ has a discontinuity of first kind at $x=1$. Discontinuity of first kind at $x=c$ is also call jump discontinuity at $x=c$

and the jump is defined as $\lim_{x\to c+} f(x) - \lim_{x\to c-} f(x)$

Here the jump at $x=1$ is $4-1=3$.

3. Discontinuity of the second kind

$f(x)$ is said to have ~~a~~ discontinuity of the second kind at $x=c$ if at least one of $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ does not exist. In this case $f$ is discontinuous whether $f$ is defined at $c$ or not.

For the function $f(x) = \frac{1}{x}$, $x=0$ is a discontinuity of the second kind.

Monotonic functions

Let $I \subset \mathbb{R}$ be an interval. A function $f: I \to \mathbb{R}$ is said to be monotonic increasing on $I$ if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$

$f: I \to \mathbb{R}$ is said to be monotonic decreasing on $I$ if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

Examples: 1. Let $f(x) = 1-x$, $x \in \mathbb{R}$

Let $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$. So, $-x_1 > -x_2$. or, $1-x_1 > 1-x_2$. So, $f(x_1) > f(x_2)$. Hence $f$ is monotonic decreasing on $\mathbb{R}$

2. Let $f(x) = x^2$, $x \in \mathbb{R}$

Let $x_1, x_2 \in \mathbb{R}$ and $0 \le x_1 < x_2$ then $f(x_1) < f(x_2)$

and $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2 \le 0$ then $f(x_1) > f(x_2)$

For example $0 \le 2 < 3$, so, $2^2 < 3^2$ so, $f(2) < f(3)$

and $-4 < -2 \le 0$, so, $(-4)^2 = 4^2 > (-2)^2 = 2^2$, so $f(-4) > f(-2)$

So, Here, $f$ is monotone increasing in $[0, \infty)$ and decreasing in $(-\infty, 0)$

A function $f: I \to \mathbb{R}$ is said to be strictly monotonic increasing on $I$ if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

$f: I \to \mathbb{R}$ is said to be strictly monotonic decreasing on $I$ if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

A function $f: I \to \mathbb{R}$ is said to be monotone or strictly monotone ~~on~~ on $I$ if either $f$ is monotonic increasing or strictly monotonic increasing or if $f$ is monotic decreasing or strictly monotonic decreasing on $I$

We now state the theorem for existence of inverse ~~the~~ function of a strictly monotonic function and its continuity

<u>Theorem 9</u> Let $I = [a, b]$ be closed and bounded interval and $f: I \to \mathbb{R}$ be strictly monotone and continuous on $I$. Then there exists an inverse function $g: J \to \mathbb{R}$ where $J = f(I)$, such that (i) $g$ is strictly increasing on $J$ and (ii) $g$ is continuous on $J$.

Examples: The exponential function and its inverse

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$, $x \in \mathbb{R}$. The range of $f$ is $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. $f$ is continuous and strictly monotic increasing on $\mathbb{R}$. Hence there exists a function $g: \mathbb{R}^+ \to \mathbb{R}$ such that $g$ is continuous and strictly monotonic increasing on $\mathbb{R}^+$ $g$ is defined by $g(y) = \log_e y$, $y \in \mathbb{R}^+$. The range of $g$ is $\mathbb{R}$

$g \circ f(x) = g(f(x)) = g(e^x) = \log_e e^x = x$ and $f \circ g(y) = f(g(y))$ $= f(\log_e y) = e^{\log_e y} = y$, $g$ is called the logarithm function.