**Exercises**: 1. Use Lagrange's multiplier method to find the shortest distance between $(-1,4)$ and the straight line $12x-5y+71=0$

2. Use Lagrange's method of multiplier to find the minimum value of $x^2+y^2+z^2$ subject to the condition $x+y+z=3$

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**Integers**: Principle of Mathematical Induction:

Let $S$ be a subset of $\mathbb{N}$ ($\mathbb{N}$ is the set of all positive integers or natural numbers) with the properties

(i) $1 \in S$

(ii) when $k \in S$, $k+1 \in S$

Then $S = \mathbb{N}$.

**Another form of Mathematical Induction**: Let $P(n)$ be a mathematical statement about positive integer $n$. Suppose

(i) $P(1)$ is true

(ii) $P(k)$ is true $\Rightarrow P(k+1)$ is true

Then $P(n)$ is true for all positive integer.

**Example 1** Show by the principle of mathematical induction

$$1+2+\cdots+n = \frac{n(n+1)}{2} \text{ for any positive integer } n$$

Proof: Let $P(n)$ be the statement

$$1+2+\cdots+n = \frac{n(n+1)}{2} \text{ for any positive integer } n$$

Now as $1 = \frac{1(1+1)}{2}$, so $P(1)$ is true

Let $P(k)$ be true. Then $1+2+\cdots+k = \frac{k(k+1)}{2}$ ($k$ is a positive integer)

So, $1+2+\cdots+k+k+1 = \frac{k(k+1)}{2} + k+1 = (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2}$
$= \frac{(k+1)(k+1+1)}{2}$

Hence $P(k+1)$ is true. So $P(n)$ is true for any positive integer $n$.

So, $1+2+\cdots+n = \frac{n(n+1)}{2}$ for any positive integer.

**Example 2** Prove that $3^{2n} - 8n - 1$ is divisible by 64 for all $n \in \mathbb{N}$

Proof: We use the principle of mathematical induction to prove the statement. Let $f(n) = 3^{2n} - 8n - 1$

Step 1 $f(1) = 9 - 8 - 1 = 0$. So, $f(1)$ is divisible by 64. So, the statement is true for $n = 1$

Step 2 $f(k+1) - f(k) = \left[3^{2k+2} - 8(k+1) - 1\right] - \left[3^{2k} - 8k - 1\right]$

$$= 8(3^{2k} - 1) = 8(9^{k} - 1)$$

$$= 8.8(9^{k-1} + 9^{k-2} + \dots + 1)$$

$$= 64p \text{ where } p \text{ is an integer.}$$

Therefore $f(k+1)$ is divisible by 64 if $f(k)$ is so.

This shows that the statement is true for $k+1$ when it is true for $k$. So, by principle of mathematical induction, the statement is true for all natural numbers $n$.

So, $3^{2n} - 8n - 1$ is divisible by 64 for all $n \in \mathbb{N}$.

**Example 3** Prove that $n! > 2^n$ for all natural number $n \geq 4$

Proof: Let $P(n)$ be the statement $n! > 2^n$ ~~for all natural numbers $n \geq 4$~~

Here $P(4)$ is true as $4! > 2^4$

Let us assume $P(k)$ is true when $k$ is a natural number and $k \geq 4$

Then $k! > 2^k$. So, $(k+1)! > 2^k (k+1) > 2^{k+1}$ as $k+1 > 2$

This shows that $P(k+1)$ is true when $P(k)$ is true

So, the statement $P(n)$ is true for all natural number $n \geq 4$

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**Exercises: 1.** Use principle of mathematical induction to prove that

(i) $1 + 3 + 5 + \dots + (2n-1) = n^2$ for all $n \in \mathbb{N}$

(ii) $3^{2n-1} + 2^{n+1}$ is divisible by 7 for all $n \in \mathbb{N}$

(iii) $10^{n+1} + 10^{n} + 1$ is divisible by 3 for all $n \in \mathbb{N}$

**Strong form of Mathematical induction or Second Principle of Mathematical Induction:** Let $S$ be a subset of $\mathbb{N}$ such that

(i) $1 \in S$

(ii) if $\{1, 2, \ldots, k\} \subset S$ then $k+1 \in S$

Then $S = \mathbb{N}$

Another form: Let $P(n)$ be a mathematical statement about positive integer $n$. Suppose

(i) $P(1)$ is true

(ii) If $P(1), P(2), \ldots, P(k)$ are true then $P(k+1)$ is true

Then $P(n)$ is true for all positive integer $n$

Example 1. Use strong principle of mathematical induction to prove that $(2+\sqrt{3})^n + (2-\sqrt{3})^n$ is an even integer for all $n \in \mathbb{N}$

Proof: Let $P(n)$ be the statement: $(2+\sqrt{3})^n + (2-\sqrt{3})^n$ is an even integer. The statement $P(1)$ is true, since $(2+\sqrt{3})^1 + (2-\sqrt{3})^1 = 4$ and it is an even integer.

Let us assume that $P(n)$ is true for $n = 1, 2, \ldots, k$,

now $(2+\sqrt{3})^{k+1} + (2-\sqrt{3})^{k+1}$

$= a^{k+1} + b^{k+1}$ where $a = 2+\sqrt{3}$ and $b = 2-\sqrt{3}$

$= (a^k + b^k)(a+b) - (a^{k-1} + b^{k-1})ab$

$= 4(a^k + b^k) - (a^{k-1} + b^{k-1})$ as $a+b = 4$ and $ab = 1$

This is an even integer, since $a^k + b^k$ and $a^{k-1} + b^{k-1}$ are even integers, by assumption.

This shows that $P(k+1)$ is true when $P(1), P(2), \ldots P(k)$ are true

So, by strong form of induction $P(n)$ is true for all $n \in \mathbb{N}$.

Exercise: 1. Use strong form of induction to prove that $(3+\sqrt{7})^n + (3-\sqrt{7})^n$ is an even integer for all $n \in \mathbb{N}$.

Well ordering principle of $\mathbb{N}$: Every non-empty subset of $\mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers (or positive integers)) has a least element.

Theorem 5: Division Algorithm:

Given integers $a$ and $b$ with $b > 0$, there exist unique integers $q$ and $r$ such that $a = bq + r$, where $0 \leq r < b$

Proof: Let us consider the subset $S$ of the set of all integers $\mathbb{Z}$ given by $S = \{a - bx : x \in \mathbb{Z},\ a - bx \geq 0\}$

First we show that $S$ is non-empty.

Since $b \geq 1$ (as $b$ is a positive integer), $|a|b \geq |a|$.

So, $a + |a|b \geq a + |a| \geq 0$ as $a + |a| = 2a \geq 0$ for $a \geq 0$ and

$a + |a| = a - a = 0$ for $a < 0$

So, $a + |a|b \geq 0 \Rightarrow a - b(-|a|) \geq 0$ and $-|a| \in \mathbb{Z}$

So, $a + |a|b \in S$ and therefore $S$ is non-empty.

Since $S$ is non-empty of non-negative integers, either

(i) $S$ contains $0$ as its least element, or

(ii) $S$ contains a smallest positive integer as its least element by the well ordering principle of $\mathbb{N}$.

In either case, we call it $r$. So, there exists an integer $q$ such that $a - bq = r$, $r \geq 0$. We prove that $r < b$. If $r \geq b$, Then $a - (q+1)b = a - qb - b = r - b \geq 0$. This shows that $a - (q+1)b \in S$ and also $a - (q+1)b = r - b < r$. This leads to a contradiction to the fact that $r$ is the least element in $S$. Hence $r < b$ and consequently, $a = bq + r$, $0 \leq r < b$. We now prove the uniqueness of $q$ and $r$. Let us suppose that $a$ has two representations: $a = bq_1 + r_1$, $a = bq_2 + r_2$, where $0 \leq r_1 < b$ and $0 \leq r_2 < b$. Then $b(q_1 - q_2) = r_2 - r_1$ or, $b|q_1 - q_2| = |r_1 - r_2|$ (as $b > 0$). But $0 \leq r_1 < b$ and $-b < -r_2 \leq 0$ give $-b < r_1 - r_2 < b$ or, $|r_1 - r_2| < b$. Consequently, $|q_1 - q_2| < 1$. Since $q_1$ and $q_2$ are integers, the only possibility is $q_1 = q_2$ and therefore, $r_1 = r_2$. This completes the proof.