Two-Element Boolean Algebra

In this section we consider a Boolean Algebra B with only two elements. Because every Boolean Algebra contains 1 and 0 and $1 \neq 0$. We find that $B = \{0, 1\}$ is a Boolean Algebra, where the operations $+$, $\cdot$, and $'$ are defined on B as follows:

$+$	1	0
1	1	1
0	1	0

$\cdot$	1	0
1	1	0
0	0	0

	$'$
1	0
0	1

In other words $1+1=1$, $1+0=1$, $0+1=1$, $0+0=0$,

$1.1=1$, $1.0=0$, $0.1=0$, $0.0=0$,

$1'=0$, $0'=1$

This B is called a two-element Boolean Algebra.

Definition Any literal symbol such as $x, y, z, x_1, x_2, \ldots, x_n$ used to represent an element of $B=\{0,1\}$ is called a Boolean Variable.

If x_1, x_2 be two independent Boolean variables, then (x_1, x_2), an ordered pair of independent Boolean variables takes values in $B \times B$. The possible ordered pairs assumed by (x_1, x_2) are $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. They are $2^2 = 4$ in numbers

Similarly (x_1, x_2, x_3) assumes the values (which are ordered triplets) $(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)$. They are $2^3 = 8$ in numbers

They are elements of $B \times B \times B = B^3$.

So, for n independent variables $x_1, x_2, \ldots, x_n$

we can define n-tuples $(x_1, x_2, \ldots x_n)$ which assumes values in $B^n = B \times B \times \ldots \times B$ (n times). $(x_1, x_2, \ldots, x_n)$ can assume 2^n values.

<u>Definition</u> A function $f: B^n \to B$ is said to be a Boolean function or expression of n variables. There are 2^{2^n} Boolean functions of n variables.

Let $f_1: B \to B$ defined by $f_1(0) = 0$ and $f(1) = 0$. It is a Boolean function in one variable. The number of Boolean function of one variable is 4.

Let $f_2: B^2 \to B$ defined by
$f_2(0,0) = 0$, $f_2(0,1) = 0$ $f_2(1,0) = 0$
$f_2(1,1) = 1$. So, it a Boolean function in two variables. There are $2^{2^2} = 16$ Boolean functions in two variables

<u>Examples</u> 1. Let $f: B^2 \to B$ defined by

$$f(x_1, x_2) = x_1 + x_2'$$

Then
$$f(0,0) = 0 + 1 = 1$$
$$f(0,1) = 0 + 0 = 0$$
$$f(1,0) = 1 + 1 = 1$$
$$f(1,1) = 1 + 0 = 1$$

2. Let $f : B^3 \to B$ be the Boolean function defined by $f(x_1, x_2, x_3) = x_1 + x_2' + x_3$

Then $f(0,0,0) = 0+1+0 = 1$ $\quad f(1,0,0) = 1+1+0 = 1$
$f(0,0,1) = 0+1+1 = 1$ $\quad f(1,0,1) = 1+1+1 = 1$
$f(0,1,0) = 0+0+0 = 0$ $\quad f(1,1,0) = 1+0+0 = 1$
$f(0,1,1) = 0+0+1 = 1$ $\quad f(1,1,1) = 1+0+1 = 1$

f assumes 0 at only one point $(0,1,0)$ of the domain and f assumes 1 at every other points of the domain.

So, for a Boolean function $f : B^n \to B$, we can construct a table called value table or assignment table or truth table, showing the values of $f(x_1, x_2, \ldots, x_n)$ for all possible values of $(x_1, x_2, \ldots, x_n)$. The following example illustrates this

Let $f : B^3 \to B$ defined $f(x_1, x_2, x_3) = x_1 + (x_2 \cdot x_3')$

~~as following~~: Then we construct the truth table as follows:

x_1	x_2	x_3	x_3'	$x_2 \cdot x_3'$	$x_1 + (x_2 \cdot x_3') = f(x_1, x_2, x_3)$
1	1	1	0	0	1
1	1	0	1	1	1
1	0	1	0	0	1
1	0	0	1	0	1
0	1	1	0	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	1	0	0

Definition: Two ~~Bool~~ Boolean function f and g defined by $f : B^n \to B$ and $g : B^n \to B$ are said to be equal if their truth tables are same and we write $f = g$.

For example, we show that $x_1 + x_2 = x_2 + x_1$

we form the truth table for $x_1 + x_2$ and $x_2 + x_1$ as follows

x_1	x_2	x_1+x_2	x_2+x_1
1	1	1	1
1	0	1	1
0	1	1	1
0	0	0	0

As truth table for $x_1 + x_2$ and $x_2 + x_1$ are same

So, $x_1 + x_2 = x_2 + x_1$

<u>Another example</u> : Let $f(x_1, x_2) = (x_1 + x_2)'$

and $g(x_1, x_2) = x_1' \cdot x_2'$

Truth table for $f(x_1, x_2)$

x_1	x_2	x_1+x_2	$(x_1+x_2)'$
1	1	1	0
1	0	1	0
0	1	1	0
0	0	0	1

Truth table for $g(x_1, x_2)$

x_1	x_2	x_1'	x_2'	$x_1' \cdot x_2'$
1	1	0	0	0
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

So, truth table of $f(x_1, x_2)$ and $g(x_1, x_2)$ are same so, $f(x_1, x_2) = g(x_1, x_2)$

or, $(x_1 + x_2)' = x_1' \cdot x_2'$