Department of Mathematics, GGGDC GE1 (SB)

Notes on GE1 (Marked portions of the syllabus) by SB (Subhasundar Bandyopadhyay)

Books followed: 1. Differential Calculus - Maity & Ghosh

2. Introduction to Discrete Mathematics - M.K. Sen & B.C. Chakraborty

Statement of Rolle's Theorem: Let $f : [a,b] \to \mathbb{R}$ be a function such that

(i) f is continuous in $[a,b]$

(ii) f is derivable in (a,b)

(iii) $f(a) = f(b)$

Then there exists at least one c such that $a < c < b$ and $f'(c) = 0$

[Note: 1. Here $\mathbb{R}$ is the set of all real numbers. $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ and $(a,b) = \{x \in \mathbb{R} : a < x < b\}$

2. f is continuous in $[a,b] \Rightarrow f$ is continuous at x such that $a < x < b$ and f is right continuous at a and left continuous at b.

3. f is derivable in $(a,b) \Rightarrow f'(x)$ exists for any x such that $a < x < b$.]

Geometrical interpretation of Rolle's theorem: If the graph $y = f(x)$ has the ordinates at two points A, B equal (A is the point on $y = f(x)$ at $x = a$ and B is the point on $y = f(x)$ at $x = b$) and if the graph of $y = f(x)$ be continuous through the interval $[a,b]$, i.e., from A to B and if the curve has a tangent at every point on it from A to B except possibly at the two extreme points A and B, then there must exist at least one point on the curve between A and B, where the tangent is parallel to the x-axis as shown by Figure 1 and Figure 2

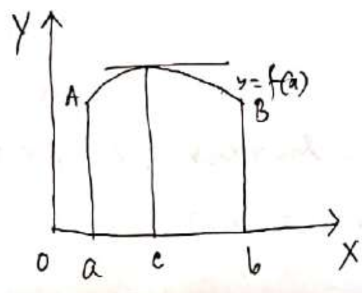

Figure 1

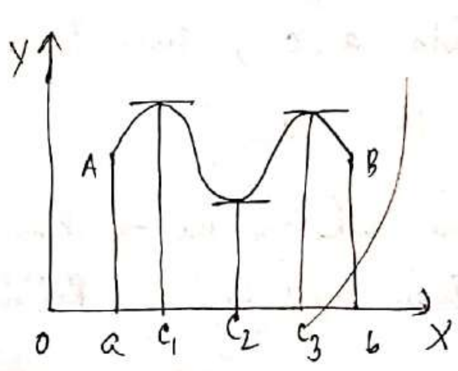

Figure 2

Example: Let $f(x) = x\sqrt{a^2-x^2}$ in $[0,a]$ $(a>0)$

Here (i) f is continuous in $[0,a]$

(ii) $f'(x) = \dfrac{x(-2x)}{2\sqrt{a^2-x^2}} + \sqrt{a^2-x^2} = \dfrac{a^2-2x^2}{\sqrt{a^2-x^2}}$ in $(0,a)$

(iii) $f(0) = f(a) = 0$

So, $f : [0,a] \to \mathbb{R}$ satisfy all the three conditions of Rolle's Theorem. So, there exists a c such that $a<c<b$ and $f'(c) = 0$ Here $c = a/\sqrt{2}$

Note: The three conditions of Rolle's Theorem are a set of sufficient conditions but they are not necessary. This means that if the three conditions are satisfied for a function $f : [a,b] \to \mathbb{R}$, then there exists a c such that $a<c<b$ and $f'(c) = 0$. But the converse is not true, i.e if there exists a point c, such that $a<c<b$ and $f'(c) = 0$ but, all the three conditions of Rolle's Theorem may not be satisfied.

Example: Let $f(x) = \frac{1}{x} + \frac{1}{1-x}$ in $[0,1]$

Here (i) f is continuous in $(0,1)$ [not in $[0,1]$]

(ii) $f'(x) = \dfrac{1}{(1-x)^2} - \dfrac{1}{x^2}$ exists in $(0,1)$

(iii) $f(0) \neq f(1)$ both being undefined. So, all three conditions of Rolle's Theorem are not satisfied. But yet there exists a c, such that $0<c<1$ such that $f'(c) = 0$. Here $c = \frac{1}{2}$

Exercise: Determine whether the function $f(x) = \sin x \cos x$, $0 \le x \le \pi/2$ satisfies the three conditions of Rolle's Theorem or not.

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Lagrange's Mean Value Theorem: Let $f : [a,b] \to \mathbb{R}$ be a function such that
(i) f is continuous in $[a,b]$
(ii) f is derivable in (a,b).
Then there exists at least one c, such that $a<c<b$ and

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof: If $f(a) = f(b)$, the theorem reduces to Rolle's Theorem. We, therefore, assume that $f(a) \neq f(b)$. Now construct the function $\phi : [a,b] \to \mathbb{R}$ given by $\phi(x) = f(x) + Ax$ where A is constant. We choose A in such a manner that $\phi(a) = \phi(b)$. So, then $f(a) + Aa = f(b) + Ab$ or, $A = -\frac{f(b)-f(a)}{b-a}$

Now, as $\phi(x) = f(x) + Ax$, $\phi'(x) = f'(x) + A$ --- (1)

Now, as f is continuous in $[a,b]$, ϕ is continuous in $[a,b]$.
As f is derivable in (a,b), ϕ is derivable in (a,b)
Also by the choice of A, $\phi(a) = \phi(b)$. So, ϕ satisfies all the conditions of Rolle's Theorem. So, there exists a point c such that $a<c<b$ and $\phi'(c) = 0$

So, $f'(c) + A = 0$ from (1)

Or, $f'(c) = -A$ or, $f'(c) = \frac{f(b)-f(a)}{b-a}$ (Putting the value of A)

Hence, the theorem is proved.

Note 1 If $b = a+h$ $(h>0)$, we can write $c = a + \theta h$, $0<\theta<1$ and consequently Lagrange's Mean Value Theorem takes the form

$$f(a+h) = f(a) + h f'(a+\theta h) \quad \text{for } 0<\theta<1$$

Note 2 Also we can write $f(h+x) - f(h) = x f'(h+\theta x)$ $(0<\theta<1)$ in $[h, h+x]$, $x>0$

Note 3. $f(x) = f(0) + x f'(\theta x)$ in $[0,x]$ $(0<\theta<1)$

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Geometrical Interpretation of Lagrange's Mean Value Theorem: If we draw the graph of $y=f(x)$ between $A(a, f(a))$ and $B(b, f(b))$ and the graph is continuous there and tangent at every point on it exist except possibly at A and B. Then there exists a point on the curve where the tangent is parallel to the chord joining A and B as shown in Figure 3

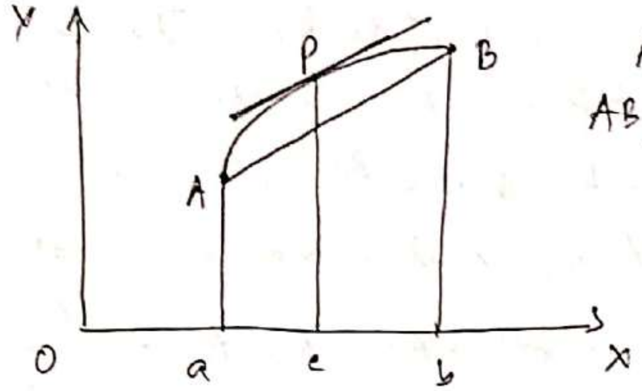

Figure-3

At P the tangent is parallel to AB, i.e., $f'(c) = \dfrac{f(b)-f(a)}{b-a}$
(Their gradient should be same)

Corollary 1: If $f'(x)=0$ in $[a,b]$ then f is constant in $[a,b]$

Proof: Let x_1, x_2 be two points such that $a \le x_1 < x_2 \le b$. Since f satisfies all the conditions of Lagrange's Mean Value Theorem in $[x_1, x_2]$, we have a point c such that $x_1 < c < x_2$ and

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \quad \text{or,} \quad f(x_2)-f(x_1) = (x_2-x_1) f'(c)$$

As $f'(c)=0$, so, $f(x_2)=f(x_1)$

So, $f(x)$ has the same value at all points between a and b

So, $f(x) =$ constant for x in $[a,b]$

Exercise: If $f'(x) = g'(x)$ in $[a,b]$ then prove that $f(x) = g(x) +$ constant in $[a,b]$

Corollary 2. If f be continuous in $[a,b]$ and $f'(x)>0$ in (a,b), then f is strictly increasing in $[a,b]$.

Proof: If x_1, x_2 are chosen so that $a \le x_1 < x_2 \le b$, by Lagrange's Mean Value Theorem $f(x_2)-f(x_1) = (x_2-x_1) f'(c)$ for some c in $a<c<b$

So, $f(x_2)-f(x_1) > 0$ as $(x_2-x_1)>0$ and $f'(c)>0$. So, for $x_1<x_2$ we have $f(x_2)>f(x_1)$. So, f is strictly increasing in $[a,b]$