Definition: ~~Toth~~ Let α be a Boolean function or expression. A Boolean expression β is said to be a dual of α if β is obtained from α by replacing each occurence of Boolean sum by Boolean product, each occurence of Boolean product by Boolean sum, each occurence of 1 by 0 and each occurence of 0 by 1. If β is a dual of α, we write $\beta = \alpha^d$

Example: Consider the Boolean expression $\alpha = (x_1 + 1)\cdot(x_2 + x_3)'$.

Then $\alpha^d = (x_1 \cdot 0) + (x_2 \cdot x_3)'$

We can prove that if $\alpha = \beta$ for any two Boolean expression α, β then $\alpha^d = \beta^d$. This is known as duality principle.

Worked out Exercise

1. Show that the Boolean expressions or functions $(x_1 \cdot x_2)\cdot x_3$ and $x_1\cdot(x_2 \cdot x_3)$ are equal

Solution: Let $\alpha(x_1, x_2, x_3) = (x_1\cdot x_2)\cdot x_3$ and $\beta(x_1,x_2,x_3) = x_1\cdot(x_2\cdot x_3)$

Consider the following truth table

x_1	x_2	x_3	$x_2\cdot x_3$	$x_1\cdot x_2$	$(x_1\cdot x_2)\cdot x_3$	$x_1\cdot(x_2\cdot x_3)$
1	1	1	1	1	1	1
1	1	0	0	1	0	0
1	0	1	0	0	0	0
1	0	0	0	0	0	0
0	1	1	1	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	0	0	0	0

Hence $\alpha(x_1, x_2, x_3) = \beta(x_1, x_2, x_3)$

So, $(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$

Definition A Boolean expression or function $\alpha(x_1, x_2, \ldots x_n)$ is said be a minterm in the variables $x_1, x_2, \ldots, x_n$ if it is of the form $\tilde{x}_1 \tilde{x}_2 \cdots \tilde{x}_n$

where each $\tilde{x}_i$ is either x_i or x_i'

Example: The Boolean functions xyz, $x'yz'$, $x'yz$ are minterms in the variables x, y, z.

Definition: A Boolean function or expression $\alpha(x_1, x_2, \ldots, x_n)$ is said to be in disjunctive normal form (DNF) or sum-of-product form in the variable $x_1, x_2, \ldots, x_n$ if there are distinct minterms $\alpha_1, \alpha_2, \ldots, \alpha_m$ in the variables $x_1, x_2, \ldots, x_n$ such that

$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m$

Note: Some authors refer this definition of a DNF as Full DNF.

Example: $\alpha(x_1, x_2, x_3) = x_1 x_2 x_3 + x_1 x_2' x_3 + x_1 x_2' x_3' + x_1' x_2 x_3$ is a boolean expression in disjunctive normal form in the variables x_1, x_2, x_3. The Boolean expressions $x + x'$, $x'y$, $xyz + x'yz + x'yz'$, in one, two and three variables respectively are in DNF whereas $(x+y)z$,

$(xy' + xz')' + x'$ are not in DNF.

We state a theorem without proof:

<u>Theorem</u> Let ~~$\alpha(x_1, x_2, x_3)$~~ α be a Boolean function or expression in the variables $x_1, x_2, \dots, x_n$ such that $\alpha \neq 0$. Then there exist minterms $\alpha_1, \alpha_2, \dots, \alpha_m$ in the variables $x_1, x_2, \dots, x_n$ such that

$$\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_m$$

We give example and show how to find DNF of a Boolean expression

Consider the Boolean expression $\alpha(x_1, x_2, x_3) = (x_1 + x_3) x_2'$.

We want to express $\alpha(x_1, x_2, x_3)$ in DNF. For this we construct its truth table

Row	x_1	x_2	x_3	x_2'	$x_1 + x_3$	$\alpha = (x_1 + x_3) x_2'$
1	1	1	1	0	1	0
2	1	1	0	0	1	0
3	1	0	1	1	1	1
4	1	0	0	1	1	1
5	0	1	1	0	1	0
6	0	1	0	0	0	0
7	0	0	1	1	1	1
8	0	0	0	1	0	0

Note that
$$\alpha(1,0,1) = 1 \quad \text{(row 3)}$$
$$\alpha(1,0,0) = 1 \quad \text{(row 4)}$$
$$\alpha(0,0,1) = 1 \quad \text{(row 7)}$$

Corresponding to row 3 $\alpha_1(x_1, x_2, x_3) = x_1 x_2' x_3$ (1 for original variable and 0 for complement)

$\alpha_2(x_1, x_2, x_3) = x_1 x_2' x_3'$ corresponding to row 4

$\alpha_3(x_1, x_2, x_3) = x_1' x_2' x_3$ corresponding to row 7

So, $x_1 x_2' x_3 + x_1 x_2' x_3' + x_1' x_2' x_3$ is the DNF

of the Boolean expression $(x_1 + x_3) x_2'$

<u>Alternative method</u>

$\alpha = (x_1 + x_3) x_2'$

$= x_1 x_2' + x_3 x_2'$ (by Distributive Law)

$= x_1 x_2' \cdot 1 + x_3 x_2' \cdot 1$ (because $\alpha \cdot 1 = \alpha$ for any Boolean expression α.)

$= x_1 x_2' (x_3 + x_3') + x_3 x_2' (x_1 + x_1')$ (because $(x_1 + x_1') = (x_3 + x_3') = 1$

$= x_1 x_2' x_3 + x_1 x_2' x_3' + x_3 x_2' x_1 + x_3 x_2' x_1'$ (by Distributive Law)

$= x_1 x_2' x_3 + x_1 x_2' x_3' + x_3 x_2' x_1'$ (because $x_1 x_2' x_3 + x_3 x_2' x_1 = x_1 x_2' x_3$)

2. Express the Boolean expression

$(x+y+z)(xy + x'z)'$ in DNF in the variable

x, y, z.

Solution: We construct the truth tables

for the Boolean expression

$\alpha = (x+y+z)(xy + x'z)'$ as follows: