Corollary 3. If f be continuous in $[a,b]$ and $f'(x)>0$ in (a,b), then f is strictly decreasing in $[a,b]$

Proof: Proof is similar to Corollary 3.

Cauchy's Mean Value Theorem: Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be two functions such that

(i) f and g are continuous in $[a,b]$

(ii) f and g are derivable in (a,b)

(iii) $g'(x)\neq 0$ for any $x\in(a,b)$

Then there exists a real number c such that $a<c<b$ and

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof: Construct the function $\phi:[a,b]\to\mathbb{R}$ by $\phi(x) = f(x)+Ag(x)$ where A is a constant. We choose A in such a way that $\phi(a)=\phi(b)$

So, $f(a)+Ag(a) = f(b)+Ag(b) \Rightarrow A = -\dfrac{f(b)-f(a)}{g(b)-g(a)}$ which exists as $g(b)-g(a)\neq 0$, since if $g(b)-g(a)=0$ or $g(b)=g(a)$ then g would satisfy all conditions of Rolle's Theorem and we would get some point d such that $a<d<b$ and $g'(d)=0$. But this is not possible as $g'(x)\neq 0$ for all $x\in(a,b)$.

Now as $\phi(x) = f(x)+Ag(x)$, $\phi'(x) = f'(x)+Ag'(x)$

Now since ϕ satisfies all the conditions of Rolle's Theorem, there exists a real number c such that $a<c<b$ and

$\phi'(c)=0$ or, $f'(c)+Ag'(c)=0$

or, $\dfrac{f'(c)}{g'(c)} = -A$ or, $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Hence, the theorem.

Note: When $g(x)=x$, Lagrange's Mean Value Theorem is a particular case of Cauchy's Mean Value Theorem.

Some problems : 1. Show that $x > \sin x$ for $0 < x < \pi/2$

Solution : Consider $f(x) = x - \sin x$, then $f'(x) = 1 - \cos x > 0$ in $0 < x < \pi/2$

Also, at ~~x=0~~, $f(0) = 0$. So, f is an increasing function and $f(x) > f(0)$ for $0 < x < \pi/2$

So, $x - \sin x > 0$ for $0 < x < \pi/2$

or, $x > \sin x$ for $0 < x < \pi/2$

2. Is Mean Value Theorem valid for $f(x) = x^2 + 3x + 2$ in $1 \le x \le 2$? Find c such that $f'(c) = 0$, if the Theorem is applicable.

Solution : Here $f(x) = x^2 + 3x + 2$, so, $f'(x) = 2x + 3$ exists in $1 < x < 2$. Also f is continuous in ~~1≤x≤2~~ $[1, 2]$

Hence Mean Value Theorem (M.V.T) is applicable. Using the Theorem, There exists c such that $f'(c) = \frac{f(2) - f(1)}{2-1} = \frac{12-6}{1} = 6$

So, $2c + 3 = 6$ or, $c = \frac{3}{2}$

3. In the M.V.T, $f(h) = f(0) + h f'(\theta h)$, $0 < \theta < 1$, show that $\lim_{h \to 0+} \theta = \frac{1}{2}$ when $f(x) = \cos x$

Solution : Since $f(x) = \cos x$, $f'(x) = -\sin x$. Hence the given relation becomes

$$\cos h = 1 - h \sin \theta h, \quad 0 < \theta < 1$$

$$\therefore \sin \theta h = \frac{1 - \cos h}{h}$$

$$\text{or, } \theta h \cdot \frac{\sin \theta h}{\theta h} = \frac{1 - \cos h}{h}$$

$$\text{or, } \lim_{h \to 0+} \theta \cdot \frac{\sin \theta h}{\theta h} = \lim_{h \to 0+} \frac{1 - \cos h}{h^2}$$

$$\text{or, } \lim_{h \to 0+} \theta = \frac{1}{2} \quad \left[\text{As } \lim_{x \to 0+} \frac{\sin x}{x} = 1\right]$$

4. If f and ϕ be continuous in $[a,b]$ and derivable in (a,b) then show that $\begin{vmatrix} f(a) & f(b) \\ \phi(a) & \phi(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(c) \\ \phi(a) & \phi'(c) \end{vmatrix}$ for some c such that $a<c<b$.

Proof: Let us construct the function $F(x) = \begin{vmatrix} f(a) & f(x) \\ \phi(a) & \phi(x) \end{vmatrix}$

Then $F'(x) = \begin{vmatrix} f(a) & f'(x) \\ \phi(a) & \phi'(x) \end{vmatrix}$

We observe that F is continuous in $[a,b]$ and derivable in (a,b). Hence using, M.V.T., we get c such that $a<c<b$

and $\dfrac{F(b)-F(a)}{b-a} = F'(c)$ Here $F(a) = \begin{vmatrix} f(a) & f(a) \\ \phi(a) & \phi(a) \end{vmatrix} = 0$

So, $\begin{vmatrix} f(a) & f(b) \\ \phi(a) & \phi(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f(c) \\ \phi(a) & \phi(c) \end{vmatrix}$

5. Use Mean value Theorem in making numerical approximations to $(28)^{1/3}$

Solution: In $f(x+h) = f(x) + h f'(x+\theta h)$, $0<\theta<1$, we put $x=27$ and $h=1$ and take $f(x) = x^{1/3}$. So, $f'(x) = \frac{1}{3}x^{-2/3}$

So, $(28)^{1/3} = (27)^{1/3} + \frac{1}{3}(27+\theta)^{-2/3}$

or, $(28)^{1/3} = 3 + \dfrac{1}{3(27+\theta)^{2/3}} < 3 + \dfrac{1}{3(27)^{2/3}} = 3 + \dfrac{1}{27}$

Also $(28)^{1/3} > 3$

So, $3 < (28)^{1/3} < 3 + \frac{1}{27}$

6. Use M.V.T to prove that $\sin 46^\circ \simeq \frac{1}{2}\sqrt{2}\left(1+\frac{\pi}{180}\right)$ ($\simeq$ means approximately equal)

Solution: In $f(x+h) = f(x) + h f'(x+\theta h)$, $0<\theta<1$, we take $f(x) = \sin x$, $x = 45^\circ$, $h = 1^\circ$. Then $f'(x) = \cos x$

So, $\sin 46^\circ = \sin 45^\circ + 1^\circ \cos\left(45^\circ + \theta . 1^\circ\right) \simeq \sin 45^\circ + 1^\circ \cos 45^\circ$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{\pi}{180}$$
$$= \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{180}\right)$$
$$= \frac{\sqrt{2}}{2}\left(1+\frac{\pi}{180}\right)$$

So, $\sin 46^\circ \simeq \frac{1}{2}\sqrt{2}\left(1+\frac{\pi}{180}\right)$

Exercises: 1. Determine which of the following functions satisfy the conditions of Rolle's Theorem in the given closed interval. When the required conditions are satisfied, find at least one value of x in that open interval at which $f'(x) = 0$

(a) $f(x) = x^2$, $x \in [-1, 1]$

(b) $f(x) = \sin x \cos x$, $x \in [0, \pi/2]$

(c) $f(x) = x(x+3)e^{-\frac{1}{2}x}$, $x \in [-3, 0]$

(d) $f(x) = |x|$, $x \in [-1, 1]$

(e) $f(x) = \tan x$, $x \in [0, \pi]$

(f) $f(x) = \cos\left(\frac{1}{x}\right)$ $x \in [-1, 1]$

2. Prove that $\sin x < x < \tan x$ for $0 < x < \pi/2$

3. Show that

(i) $\frac{x}{1+x} < \log(1+x) < x$, if $x > 0$

(ii) $x < \log\frac{1}{1-x} < \frac{x}{1-x}$, if $0 < x < 1$

(iii) $x^3 - 3x^2 + 3x + 2$ is increasing in any interval