Statement of Taylor's Theorem with Lagrange's form of remainder and Cauchy's form of remainder: If a function f be such that

(i) f^{n-1} be continuous in $[a, a+h]$ $(h>0)$

(ii) f^{n} exists in $(a, a+h)$

then there exists at least one number θ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

R_n is called the remainder term. When $R_n = \frac{h^n}{n!} f^n(a+\theta h)$, $0<\theta<1$, the theorem is called Taylor's Theorem with Lagrange's form of remainder.

When $R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$, $0<\theta<1$, the theorem is called Taylor's theorem with Cauchy's form of remainder.

Statement of Maclaurin's Theorem with Lagrange's form of remainder and Cauchy's form of remainder: If a function f be such that

(i) f^{n-1} be continuous in $[0, x]$ $(x>0)$

(ii) f^{n} exists in $(0, x)$ then there exists at least one number θ such that $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$. R_n is called the remainder term. When $R_n = \frac{x^n}{n!} f^n(\theta x)$, $0<\theta<1$, the theorem is called Maclaurin's theorem with Lagrange's form of remainder. When $R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$, $0<\theta<1$, the theorem is called Maclaurin's theorem with Cauchy's form of remainder.

Note: f^n is the nth order derivative of f.

Some Examples: 1. Apply Maclaurin's theorem to $f(x) = (1+x)^4$ to deduce that $(1+x)^4 = 1+4x+6x^2+4x^3+x^4$

Solution: Since f possesses derivatives of every order for every real numbers. So, by Maclaurin's Theorem with Lagrange's form of remainder after four terms,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(\theta x), \quad 0<\theta<1$$

Here $f(x) = (1+x)^4$, $f'(x) = 4(1+x)^3$, $f''(x) = 12(1+x)^2$, $f'''(x) = 24(1+x)$, $f^{IV}(x) = 24$.

So, $f(0) = 1$, $f'(0) = 4$, $f''(0) = 12$, $f'''(0) = 24$, $f^{IV}(\theta x) = 24$

Hence $f(x) = (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

2. Expand $\sin x$ in a finite series in powers of x, with remainder in Lagrange's form.

Solution: Let $f(x) = \sin x$, then $f^n(x) = \sin\left(\frac{1}{2}n\pi + x\right)$. So, $f^n(0) = \sin\frac{1}{2}n\pi$ so that f possesses derivatives of every order for every real number.

By Maclaurin's theorem with Lagrange's form of remainder, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x), \quad 0<\theta<1$$

$$\text{So, } \sin x = \sin 0 + x \sin\frac{1}{2}\pi + \frac{x^2}{2!}\sin\pi + \cdots + \frac{x^{n-1}}{(n-1)!}\sin\frac{1}{2}(n-1)\pi + \frac{x^n}{n!}\sin\left(\frac{1}{2}n\pi + \theta x\right), \quad 0<\theta<1$$

$$\text{or, } \sin x = x - \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!}\sin\frac{1}{2}(n-1)\pi + \frac{x^n}{n!}\sin\left(\frac{1}{2}n\pi + \theta x\right), \quad 0<\theta<1$$

3. Show that $\lim_{h\to 0}\theta = \frac{1}{n+1}$, where θ is given by

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h), \quad 0<\theta<1$$

provided that f^{n+1} is continuous at a, $f^{n+1}(a) \neq 0$

Solution: Since f^{n+1} is continuous at a, then $f^{n+1}(x)$ exist in $[a-\delta, a+\delta]$, for some $\delta>0$. Taking $a+h$, a point of this interval,

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h), \quad 0<\theta<1$$

$$\text{and } f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta' h),$$
$0<\theta'<1$.

Then $f^n(a+\theta h) = f^n(a) + \frac{h}{n+1} f^{n+1}(a+\theta' h)$. Using Lagrange's mean value theorem, we have,

$$f^n(a) + \theta h f^{n+1}(a+\theta\theta'' h) = f^n(a) + \frac{h}{n+1} f^{n+1}(a+\theta' h), \quad 0<\theta''<1$$

$$\text{or, } \theta f^{n+1}(a+\theta\theta'' h) = \frac{1}{n+1} f^{n+1}(a+\theta' h)$$

$$\text{or, } \lim_{h\to 0}\theta \cdot f^{n+1}(a) = \frac{1}{n+1} f^{n+1}(a), \text{ since } f^{n+1}(x) \text{ is continuous.}$$

$$\text{or, } \lim_{h\to 0}\theta = \frac{1}{n+1}, \text{ since } f^{n+1}(a) \neq 0$$

Exercises 1. Obtain the expansions of the following functions with the remainder in Lagrange's form: $[0<\theta<1]$

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} e^{\theta x}$

(ii) $\sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}(\sin\theta x + \cos\theta x)$

2. If $f(h) = f(0) + h f'(0) + \frac{h^2}{2!} f''(\theta h)$, $0<\theta<1$, then prove that $\theta = \frac{9}{25}$ when $h = 1$ and $f(x) = (1-x)^{5/2}$.

Taylor's Infinite Series : If f be a function possessing derivatives of all orders in $[a, a+h]$, then to every positive integer n, however large, there corresponds a Taylor's Theorem of the form

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n,$$ where R_n is the remainder after n term, may be taken in any form.

Writing $S_n = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$,

$$f(a+h) = S_n + R_n.$$

Now, let us suppose, in addition, that $R_n \to 0$ as $n\to\infty$, then

$$\lim_{n\to\infty} S_n = f(a+h)$$

or, $f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \cdots = f(a+h)$

This leads to

Theorem If (i) f be a function possessing derivatives of all orders in $[a, a+h]$ and (ii) remainder $R_n \to 0$ as $n\to\infty$, then Taylor's series in infinite form, i.e., $f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^n(a) + \cdots$ is convergent and converges to $f(a+h)$.

Other forms (1) Put $a+h = b$ or, $h = b-a$ then

$$f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^n}{n!} f^n(a) + \cdots = f(b)$$

(2) writing x for a, $f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n}{n!} f^n(x) + \cdots = f(x+h)$

(3) changing $a+h$ to x, i.e., h to $x-a$

$$f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^n}{n!}f^n(a)+\cdots=f(x)$$

Corollary: Maclaurin's Infinite series

When $a=0$, $h=x$, in the theorem, we observe that if (i) f possesses derivatives of all orders in $[0,x]$ and (ii) $R_n \to 0$ as $n\to\infty$, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^n}{n!}f^n(0)+\cdots$$

which is called Maclaurin's infinite series.

Some examples 1. Find the infinite series for e^x

Solution: Let $f(x)=e^x$, $f^n(x)=e^x$ and hence $f(x)$ possesses derivatives of every order for every value of x. Again $f^n(x)=e^x$. So, $f^n(0)=e^0=1$

and R_n (Lagrange's form) $=\frac{x^n}{n!}e^{\theta x}$ for $0<\theta<1$

Now, (i) $e^{\theta x}$ $(0<\theta<1)$ lies between $e^0=1$ and e^x for every value of x (as e^x is ~~an incr~~ a monotonic increasing function) and hence bounded.

(ii) Also $\left|\frac{x^n}{n!}\right|\to 0$ as $n\to\infty$

Thus $R_n\to 0$ as $n\to\infty$ and e^x is represented by

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$$ ~~which is valid for all x~~.

which is valid for all x.

2. Find the infinite series for $\sin x$

Solution: Let $f(x)=\sin x$, $f^n(x)=\sin\left(\frac{1}{2}n\pi+x\right)$, so that $\sin x$ possesses derivatives of all order for all values of x. Again

$f^n(0)=\sin\frac{1}{2}n\pi$, and R_n (Lagrange's form) $=\frac{x^n}{n!}\sin\left(\frac{1}{2}n\pi+\theta x\right)$, $0<\theta<1$

Now, $\left|\frac{x^n}{n!}\right|\to 0$ as $n\to\infty$ and $\left|\sin\left(\frac{1}{2}n\pi+\theta x\right)\right|\le 1$ and hence $R_n\to 0$ as $n\to\infty$. So, $\sin x$ is represented by

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^r\frac{x^{2r+1}}{(2r+1)!}+\cdots$$

Exercise: Find the infinite series for $\cos x$

Solution: Similar proof as 2.