3. Find the infinite series for $(1+x)^m$, m is a real number.

Solution: Let $f(x) = (1+x)^m$. Then

$$f^n(x) = m(m-1)(m-2)\cdots(m-n+1)(1+x)^{m-n}$$

$$f^n(0) = m(m-1)(m-2)\cdots(m-n+1)$$

$$R_n(\text{Cauchy}) = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x), \quad 0<\theta<1$$

$$= \frac{m(m-1)(m-2)\cdots(m-n+1)}{(n-1)!} x^n \cdot (1+\theta x)^{m-1} \cdot \left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$$

and $f^n(x)$ exists for every value of x when m is a positive integer and $f^n(x)$ exists for every value of $x>-1$, when m is a real number other than positive integer.

Now, $(1+x)^m = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n$

$$= 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\cdots(m-n+2)}{(n-1)!} + R_n$$

Case 1 If m be a positive integer, the expansion ends after $(m+1)$th term. Hence,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + x^m$$

and this valid for every x.

Case 2 If, however, m be any real ~~numbe~~ number other than positive integer, then we see that, for $|x|<1$,

(i) $\frac{m(m-1)(m-2)\cdots(m-n+1)}{(n-1)!}x^n \to 0$ as $n\to\infty$

(ii) $(1+\theta x)^{m-1}$ is finite

(iii) $\left|\frac{1-\theta}{1+\theta x}\right| < 1$ and hence $\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$ is bounded

and hence $R_n \to 0$ as $n\to\infty$. Then we get

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!}x^n + \dots$$

and this is valid for $|x|<1$

4. Find the infinite series for $\log(1+x)$

Solution: Let $f(x) = \log(1+x)$. The function $f(x) = \log(1+x)$ is defined and its derivatives of every order exist for every value of $x > -1$.

Again $f^n(x) = (-1)^{n-1}\dfrac{(n-1)!}{(1+x)^n}$

$$f^n(0) = (-1)^{n-1}(n-1)!$$

Now, $R_n(\text{Lagrange}) = \dfrac{x^n}{n!} f^n(\theta x) = \dfrac{(-1)^{n-1}}{n}\left(\dfrac{x}{1+\theta x}\right)^n, \quad 0<\theta<1$

and $R_n(\text{Cauchy}) = \dfrac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x) = (-1)^{n-1} \cdot \dfrac{1}{1+\theta x} \cdot \left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1}, \quad 0<\theta<1$

Taking Cauchy's form of remainder in $|x|<1$, we find that

(i) $x^n \to 0$ as $n \to \infty$

(ii) $\dfrac{1}{1+\theta x}$ is finite

(iii) $\left|\dfrac{1-\theta}{1+\theta x}\right| < 1$ and hence $\left|\dfrac{1-\theta}{1+\theta x}\right|^{n-1}$ is bounded

Thus, $R_n \to 0$ as $n \to \infty$ in $|x|<1$

Again when $x=1$, taking Lagrange's form of remainder we see that

$$R_n = \frac{(-1)^{n-1}}{n} \cdot \frac{1}{(1+\theta)^n} \to 0 \text{ as } n \to \infty$$

So, $$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots + (-1)^{n-1}\frac{x^n}{n} + \dots$$

which is valid when $-1 < x \leq 1$

Indeterminate Forms: L'Hospital's Rule :

The ratio $\frac{f(x)}{g(x)}$ is not defined for $x = a$ when $g(x) = 0$. If $f(a)$ and $g(a)$ be both zeroes, the ratio has the indeterminate form $\frac{0}{0}$. It is possible that $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists in this case.

The form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 0^0, ∞^0, 1^∞ are all the indeterminate forms.

~~Note~~ L'Hospital's Rule for 0/0 form: If $f(x)$ and ~~be~~ $g(x)$ be such that it is continuous in $a-h \le x \le a+h$ and derivable in $a-h < x < a+h$ and $f(a) = g(a) = 0$. ~~Also~~ Also $\lim_{x\to a} \frac{f'(x)}{g'(x)} = A$. Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = A.$$

Note: If $f(x)$ and $g(x)$ is continuous and derivable when x is sufficiently large and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ if the limit $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists.

L'Hospital's Rule for $\frac{\infty}{\infty}$ form: If $f(x)$ and $g(x)$ be two functions such that they are continuous in $a-h \le x \le a+h$ and derivable in $a-h < x < a+h$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = l$ (or $\pm\infty$) if $\lim_{x\to a} \frac{f'(x)}{g'(x)} = l$ (or $\pm\infty$)

Note: Similar is the case as $x \to \infty$

Other indeterminate forms:

(i) The form $0 \cdot \infty$ is obviously reducible to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$

(ii) The form $\infty - \infty$ is ~~read~~ readily altered to $\infty \cdot 0$ for if $f(x)$ and $\phi(x) \to 0$ as $x \to a$, we have $f - \phi = f\phi\left(\frac{1}{\phi} - \frac{1}{f}\right)$

(iii) The three exponential forms 0^0, ∞^0, $1^{\pm\infty}$ are dealt with by taking their logarithms; in all these cases these leads to $0.\infty$.

Problems

1\. Find $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x - \sin x}$ using L'Hospital's rule

Solution: $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x - \sin x} \left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0} \dfrac{\sec^2 x - 1}{1-\cos x} \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{2\sec^2 x \tan x}{\sin x} \left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0} \dfrac{4\sec^2 x \tan^2 x + 2\sec^4 x}{\cos x} = 2$ (using L'Hospital's rule)

2\. Find $\lim\limits_{x\to 0} \cot x \cdot \log\dfrac{1+x}{1-x}$

Solution: $\lim\limits_{x\to 0} \cot x \cdot \log\dfrac{1+x}{1-x}$ $(\infty \cdot 0)$

$= \lim\limits_{x\to 0} \dfrac{\log(1+x) - \log(1-x)}{\tan x} \left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0} \dfrac{\dfrac{1}{1+x} - \dfrac{1}{1-x}}{\sec^2 x} = 0$ (using L'Hospital's rule)

3\. Find $\lim\limits_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

Solution: $\lim\limits_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$ $(\infty - \infty)$

$= \lim\limits_{x\to 0} \dfrac{\sin x - x}{x\sin x} \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{\cos x - 1}{x\cos x + \sin x} \left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0} \dfrac{-\sin x}{2\cos x - x\sin x} = 0$ (using L'Hospital's rule)