4. Find $\lim\limits_{x\to 0} (\sin x)^{2\tan x}$

Solution: $\lim\limits_{x\to 0} (\sin x)^{2\tan x}$ $\left(0^0\right)$

Take $y = (\sin x)^{2\tan x}$ Then $\log y = 2\tan x \log \sin x$

Then $\lim\limits_{x\to 0} \log y = \lim\limits_{x\to 0} 2\tan x \log \sin x = \lim\limits_{x\to 0} \dfrac{2\log \sin x}{\cot x}$ $\left(-\dfrac{\infty}{\infty}\right)$

$= \lim\limits_{x\to 0} \dfrac{2\cot x}{-\operatorname{cosec}^2 x} = \lim\limits_{x\to 0} -2\sin x \cos x = 0$ (By L'Hospital's rule)

Now $\lim\limits_{x\to 0} \log y = \log \lim\limits_{x\to 0} y$ [as log is continuous function]

So, $\log \lim\limits_{x\to 0} y = 0 = \log 1$

So, $\lim\limits_{x\to 0} y = 1$

or, $\lim\limits_{x\to 0} (\sin x)^{2\tan x} = 1$

5. Find $\lim\limits_{x\to 0} \left(1 + \dfrac{1}{x}\right)^x$

Solution: $\lim\limits_{x\to 0} \left(1+\dfrac{1}{x}\right)^x$ $\left(\infty^0\right)$ $\cancel{= \lim\limits_{x\to 0}}$

Let $y = \left(1+\dfrac{1}{x}\right)^x$ So, $\log y = x \log\left(1+\dfrac{1}{x}\right) = \dfrac{\log(x+1) - \log x}{1/x}$

So, $\lim\limits_{x\to 0} \log y = \lim\limits_{x\to 0} \dfrac{\log(x+1) - \log x}{1/x}$ $\left(\dfrac{\infty}{\infty}\right)$

$= \lim\limits_{x\to 0} \dfrac{(x+1)^{-1} - x^{-1}}{-x^{-2}} = \lim\limits_{x\to 0} \left(x - \dfrac{x^2}{x+1}\right) = 0$

Now $\lim\limits_{x\to 0} \log y = \log \lim\limits_{x\to 0} y$ [as log is continuous]

So, $\lim\limits_{x\to 0} \log y = \log \lim\limits_{x\to 0} y = 0 = \log 1$ So, $\lim\limits_{x\to 0} y = 1$

So, $\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x = 1$.

6. If $\lim_{x\to 0}\frac{\sin 2x + a\sin x}{x^3}$ be finite, find the value of ~~aa~~ a and the value of the limit.

Solution: $\lim_{x\to 0}\frac{\sin 2x + a\sin x}{x^3}\left(\frac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{2\cos 2x + a\cos x}{3x^2} \quad \text{(By L'Hospital's rule)} \quad \text{--- (1)}$$

Since denominator $3x^2 \to 0$ as $x\to 0$, then ~~this~~ in order that limit of (1) be finite as $x\to 0$, it is necessary that the ~~number~~ numerator should also $\to 0$. This gives

$$2+a = 0 \quad \text{or} \quad a = -2$$

So, the required limit is $\lim_{x\to 0}\frac{2\cos 2x \cancel{+} -2\cos x}{3x^2}\left(\frac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{-4\sin 2x + 2\sin x}{6x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-8\cos 2x + 2\cos x}{6} \quad \text{(Using L'Hospital's rule)}$$

$$= \frac{-8+2}{6} = -1$$

7. Find the values of a and b such that $\lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3} = \frac{1}{3}$, assuming that L'Hospital's rule is applicable.

Solution: $\lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3}\left(\frac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{1-a\cos x + ax\sin x + b\cos x}{3x^2} \quad \text{(By L'Hospital's rule)} \quad \text{--- (1)}$$

Since denominator $3x^2 \to 0$ as $x\to 0$, then in order that the limit of

(1) should be finite as $x \to 0$, it is necessary that the ~~numer~~ numerator should also $\to 0$ as $x \to 0$. This gives $1 - a + b = 0$ ... (2)
In this the limit should be of the form $0/0$. So, Using L'Hospital's rule again, limit in (1)

$$= \lim_{x\to 0} \frac{(2a-b)\sin x + a x \cos x}{6x} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{(3a-b)\cos x - a x \sin x}{6} \text{ (using L'Hospital's rule)}$$

$$= \frac{3a-b}{6} = \frac{1}{3} \text{ (given}$$

This gives $3a - b = 2$ ... (3)

Solving (2) and (3), we have $a = \frac{1}{2}$, $b = -\frac{1}{2}$

Exercises 1. Use L'Hospital's rule to find the following limits:

(i) $\lim_{x\to 0} \frac{e^x - 1}{x}$ (ii) $\lim_{x\to 0} \frac{1-\cos 2x}{x^2}$ (iii) $\lim_{x\to 0} \frac{x - \sin x}{x^3}$

(iv) $\lim_{x\to 0} \frac{e^x + \sin x - 1}{\log(1+x)}$ (v) $\lim_{x\to\infty} \frac{\log x}{x}$

(vi) $\lim_{x\to 0} \frac{\log \sin x}{\cot x}$ (vii) $\lim_{x\to 0} x^2 \log(x^2)$

(viii) $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$ (ix) $\lim_{x\to 1} \left\{\frac{x}{x-1} - \frac{1}{\log x}\right\}$

(x) $\lim_{x\to \pi/2} (\sin x)^{\tan x}$ (xi) $\lim_{x\to 0} x^{2\sin x}$

(xii) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$ (xiii) $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

2. Determine a such that $\lim_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x}$ exists and equal to 1.

3. Find the values of a and b in order that

$$\lim_{x \to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1$$

Application of the principle of Maxima and Minima for a function of single variable in geometrical, physical and to other problems:

Example 1 Show that the maximum rectangle

(i) with a given area is a square

(ii) inscribable in a circle is a square

Show also that of all rectangles of ~~a~~ given area

(iii) the square has the smallest perimeter.

Solution: 1.(i) Let the rectangle ABCD in Figure 1 have adjacent sides $AB = x$, $AD = y$. Let the given perimeter be $2a$, where a is a constant. Then $2(x+y) = 2a$

or, $x+y = a$ and area of the rectangle $= A = xy = x(a-x) = ax - x^2$

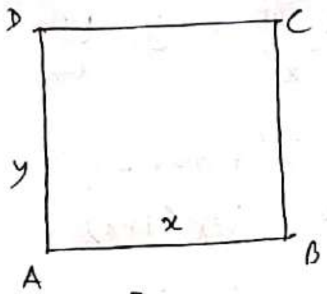

Figure 1

Then $\frac{dA}{dx} = a - 2x = 0$ when $x = a/2$

Also $\frac{d^2A}{dx^2} = -2 < 0$, So A is maximum

when $x = \frac{a}{2}$. Then $y = \frac{a}{2}$ and the rectangle becomes a square.

1 (ii) Let ABCD be the inscribed rectangle in Figure 2

Taking $OA = r$, fixed where O is the centre of the circle. Let $AB = x$, $AD = y$. Then the area of the rectangle $= A = xy$. As ABCD is a rectangle, AC is the diameter of the circle

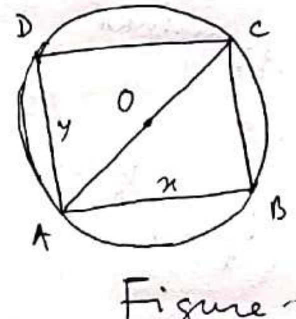

Figure 2

So, $AC = 2r$. Now $AB^2 + BC^2 = AC^2$ or, $x^2 + y^2 = 4r^2$

So, $A^2 = x^2y^2 = x^2(4r^2 - x^2)$

Now, $\dfrac{dA^2}{dx} = x^2(-2x) + 2x(4r^2 - x^2) = 2x \cdot 4r^2 - 4x^3 = 4x(2r^2 - x^2)$

So, $\dfrac{dA^2}{dx} = 0 \Rightarrow x^2 = 2r^2$ or $x = \sqrt{2}\,r$

Now $\dfrac{d^2A^2}{dx^2} = 8r^2 - 12x^2 = 8r^2 - 24r^2 = -16r^2 < 0$ at $x = \sqrt{2}\,r$

Thus A^2 and hence A, the area of the inscribed rectangle is maximum when $x = \sqrt{2}\,r$. Then $y = \sqrt{2}\,r$ and the rectangle becomes a square.

1.(iii) Let ABCD be the rectangle of given area in Figure. Let AB $= x$ and AD $= y$.

Then as area of the rectangle is fixed,

So, $xy = k$, k is fixed positive constant.

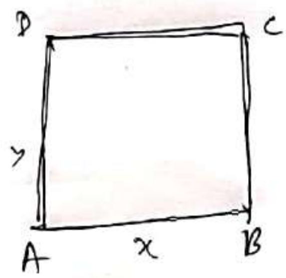

figure 3

Now perimeter of the rectangle $= P = 2(x+y)$

or, $P = 2(x+y) = 2\left(x + \dfrac{k}{x}\right)$

Now, $\dfrac{dP}{dx} = 2\left(1 - \dfrac{k}{x^2}\right)$. So, $\dfrac{dP}{dx} = 0 \Rightarrow x = \sqrt{k}$

Now $\dfrac{d^2P}{dx^2} = \dfrac{4k}{x^3} = \dfrac{4k}{k^{3/2}} > 0$ at $x = \sqrt{k}$

Hence, for smallest perimeter P, $x = \sqrt{k}$ and hence $y = \sqrt{k}$ and the rectangle reduces to a square.

Exercises: 1. Show that the largest triangle inscribable in a circle is equilateral.

2. Show that the largest triangle with a given perimeter is equilateral

3. Show that the sides of the largest rectangle that can be inscribed in a semi-circle of radius r are $\sqrt{2}\,r$ and $r/\sqrt{2}$.