Example 2 A farmer can afford to buy 800 metres of wire-fencing. He wishes to enclose a rectangular field of largest possible area. What should be the dimensions of the field be?

Solution: This is Example 1(i). Only the perimeter = 800 metres. Since the rectangle becomes a square, each side = 200 metres

Example 3 ~~A farmer~~ The intensity of light varies inversely as the square of the distance from the source. If two lights are 150 metres apart and one light is 8 times as strong as the the other, where should an ~~of~~ object be placed between the lights to have the least illumination?

Solution: Let P, Q be two lights of which P is stronger and O be the object placed between them such that PO = x. Then OQ = 150 − x. Taking I as the intensity of light,

$$I = \frac{8K}{x^2} + \frac{K}{(150-x)^2}$$, K being the constant of variation. (Here K is taken as positive)

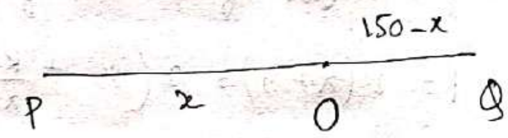



Figure 4

Now $$\frac{dI}{dx} = \frac{-16K}{x^3} + \frac{2K}{(150-x)^3}$$ and $$\frac{d^2I}{dx^2} = \frac{6K}{(150-x)^4} + \frac{48K}{x^4}$$

Now, $$\frac{dI}{dx} = 0 \Rightarrow \frac{-16K}{x^3} + \frac{2K}{(150-x)^3} = 0$$

$$\Rightarrow \left(\frac{x}{150-x}\right)^3 = 8 \Rightarrow \frac{x}{150-x} = 2 \Rightarrow x = 300 - 2x$$

$$\Rightarrow 3x = 300 \Rightarrow x = 100$$ Then $\frac{d^2I}{dx^2}$ is positive

Hence for least illumination PO = 100 metres.

Example 4 It is desired to make an open box with square base out of a square piece of cardboard of side 1 foot by cutting equal square out of the corners and then folding up the cardboard to form the sides. What must be the length of the square cut in order that the volume be a maximum?

Solution: Let $x$ = length of the side of the square to be cut from each corner (see figure 5)

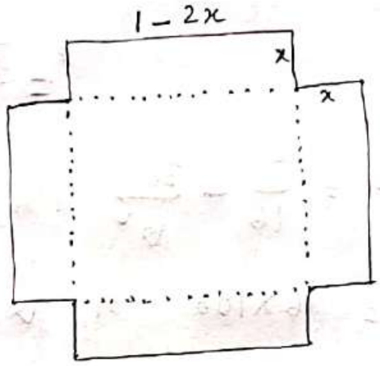


Figure 5

The volume V of the proposed box can be written as

$V = (1-2x)(1-2x)x$ cubic feet, which is a function of a single variable $x$

and $0 \le x \le \frac{1}{2}$ (As $x \geq 0$ and $1-2x \geq 0$)

Now $V = (1-2x)^2 x = (1-4x+4x^2)x = x - 4x^2 + 4x^3$

So, $\frac{dV}{dx} = 1 - 8x + 12x^2 = 12x^2 - 6x - 2x + 1 = 6x(2x-1) - (2x-1) = (6x-1)(2x-1)$

$= (1-2x)(1-6x)$

Now, $\frac{dV}{dx} = 0 \Rightarrow x = \frac{1}{2}, \frac{1}{6}$. Of these two values of $x$ we can discard $x = \frac{1}{2}$ immediately since $V = 0$ when $x = \frac{1}{2}$

Now $\frac{d^2V}{dx^2} = -8 + 24x$ So, $\frac{d^2V}{dx^2} = -8 + 24 \times \frac{1}{6} = -8+4 = -4 < 0$

So, V is maximum when $x = \frac{1}{6}$

So, maximum volume $= \frac{2}{27}$ cubic feet and side of the square cut out is $\frac{1}{6}$ foot.

Example 5 The cost of fuel in running a locomotive is proportional to the square of the speed and is Rs. 48 per hour for a speed of 16 miles per hour. Other cost amounts to Rs. 300 per hour. What is the most economical speed?

Solution: Let $v$ = required speed and $C$ = total cost per mile.
Cost of fuel per hour = $kv^2$, $k$ is a constant of proportionality.
Since $v = 16$ when cost of fuel = Rs. 48 per hour, we have

$$48 = k \cdot 16^2 \quad \text{or,} \quad k = \frac{3}{16}$$

$$\therefore C \text{ (in Rupees per mile)} = \frac{\text{cost in Rupees per hour}}{\text{speed in miles per hour}}$$

$$= \frac{\frac{3}{16}v^2 + 300}{v} = \frac{3}{16}v + \frac{300}{v}$$

So, $\frac{dC}{dv} = \frac{3}{16} - \frac{300}{v^2}$. So, $\frac{dC}{dv} = 0 \Rightarrow \frac{3}{16} = \frac{300}{v^2}$

or, $v^2 = 16 \times 100$ or, $v = 40$

and $\frac{d^2C}{dv^2} = \frac{600}{v^3}$

So, $\frac{d^2C}{dv^2} > 0$ when $v = 40$

So, 40 miles per hour for the locomotive gives the minimum cost. So, the most economical speed is 40 miles per hour.

Example 6 Divide 120 into two parts such that the product of one part and the square of the other is maximum.

Solution: Let the two parts be $x$ and $y$.

So, $x + y = 120$, Let $A = xy^2$ we have to find $x$ and $y$ such that $A$ is maximum.

Now $A = xy^2 = x(120-x)^2$ (As $x+y = 120$)

Now $\frac{dA}{dx} = -2x(120-x) + (120-x)^2 = (120-x)(120-3x)$

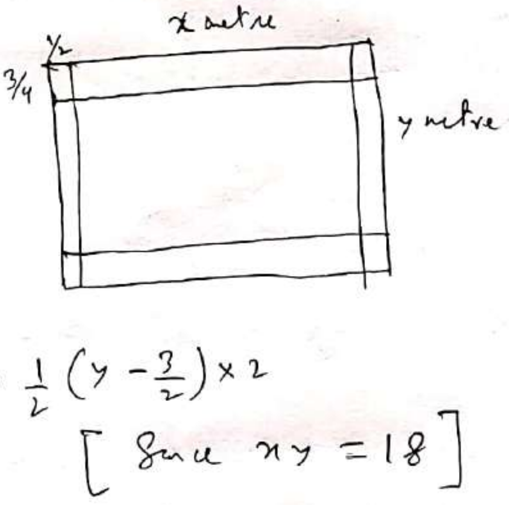
So, $\frac{dA}{dx} = 0 \Rightarrow x = 120, 40$. We can discard $x = 120$ as $A = 0$ when $x = 120$.

Now $$\frac{d^2A}{dx^2} = -2(120-x) + 2x - 2(120-x)$$
$$= -4 \times 80 + 2 \times 40 < 0 \quad \text{when } x = 40$$

So, $A$ is maximum when $x = 40$.

So, the required parts are $x = 40$ and $y = 80$

Example 7 A sheet of paper for a poster contains 18 square metres. The margin at the top and bottom are $3/4$ metres and the sides $1/2$ metres. What are the dimensions of the sheet of paper if the printed area is maximum?




Solutions: Let $x$ metre and $y$ metre be the dimensions of the sheet of paper. Let $A$ be the printed area. Then by the given condition

$$A = xy - 4 \times \frac{3}{4} \times \frac{1}{2} - \frac{3}{4}(x-1) \times 2 - \frac{1}{2}\left(y - \frac{3}{2}\right) \times 2$$
$$= 18 - \frac{3}{2} - \frac{3}{2}x + \frac{3}{2} - y + \frac{3}{2} \quad [\text{Since } xy = 18]$$
$$= 18 - \frac{3}{2}x - y + \frac{3}{2} = 18 - \frac{3}{2}x - \frac{18}{x} + \frac{3}{2} \quad [\text{As } xy = 18]$$

Now $\frac{dA}{dx} = -\frac{3}{2} + \frac{18}{x^2}$ So, $\frac{dA}{dx} = 0 \Rightarrow x^2 = 12$ or $x = 2\sqrt{3}$

$\frac{d^2A}{dx^2} = -\frac{36}{x^3} < 0$ when $x = 2\sqrt{3}$. So, the printed area is maximum when $x = 2\sqrt{3}$. So, the dimensions of the sheet of paper is $2\sqrt{3}$ and $3\sqrt{3}$ (as $xy = 18$) when the printed area is maximum.