Exercises 1. (a) Show that the largest triangle with a given perimeter is equilateral

(b) Show that the largest triangle inscribable in a circle is equilateral.

(c) Show that the sides of the largest rectangle that can be inscribed in a semi-circle of radius r are $\sqrt{2}r$ and $r/\sqrt{2}$.

(d) Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$

2. Show that the semi-vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

3. Divide 8 into two parts such that the sum of the square of the two parts may be a minimum.

4. Assuming that the ~~petro~~ petrol burnt in driving motor boat varies as the cube of its speed, show that the most economical speed when against a current c miles per hour is $\frac{3}{2}c$ miles per hour.

Maxima and minima of functions of not more than three variables

Definition A function $f(x,y)$ of two variables x and y is said to have a maximum or minimum at the point (a,b) according as $f(x,y) \le f(a,b)$ or $f(x,y) \ge f(a,b)$ in some some suitably small neighbourhood of (a,b).

Thus, for a maximum $\Delta f = f(a+h, b+k) - f(a,b) < 0$ and for a minimum $\Delta f > 0$ when h^2+k^2 is sufficiently small.

$f(a,b)$ is called an extreme value of $f(x,y)$ if it is either a maximum or a minimum.

A necessary condition for an Extreme value

We state a Theorem :

Theorem 1 If f has an extreme value at (a,b) if the partial derivatives f_x and f_y exist at (a,b), then $f_x(a,b)=0$ and $f_y(a,b)=0$.

Note: A point (a,b) is said to be a stationary point of $f(x,y)$ if $f_x(a,b) = 0 = f_y(a,b)$. A stationary point may or may not be an extreme point. We give an example where a stationary point is an extreme point

Example 1 Let $f(x,y) = 0$ when $x=0$ or $y=0$
$= 1$ elsewhere

Here

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{0}{k} = 0$$

So, $(0,0)$ is a stationary point of f. Now $f(0,0)=0$ and $f(x,y) \geq f(0,0)$ for all points (x,y) in any neighbourhood of $(0,0)$. Hence f has a minimum value at $(0,0)$

Now we give an example where a stationary point is not an extreme point

Example 2 Let $f(x,y) = (x-y)^3 + (2-x)^2$

Then $f_x(x,y) = 3(x-y)^2 - 2(2-x)$, $f_y(x,y) = -3(x-y)^2$

Now $f_x(2,2) = 0$ and $f_y(2,2) = 0$

So, $(2,2)$ is a stationary point of f. For any point $(2+h, 2+k)$ in any neighbourhood of $(2,2)$,

$$\Delta f = f(2+h, 2+k) - f(2,2) = (h-k)^3 + h^2$$

For $h=0$, $k>0$, $\Delta f < 0$ and for $h>0$, $k=0$, $\Delta f > 0$

So, $(2,2)$ is not an extreme point of f.

So, we see from example 2 that Theorem 1 is a necessary but not sufficient condition for extremum values of f.

A sufficient condition for extremum value:

Theorem 2 Let $f(x,y)$ have continuous second order partial derivatives in a certain neighbourhood of (a,b) where $f_x(a,b)=0$ and $f_y(a,b)=0$ and f_{xx}, f_{yy} and f_{xy} are not all zero at (a,b). Then

(i) f has no extreme value at (a,b) if $H = f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b)

(ii) f has an extreme value at (a,b) if $H > 0$ at (a,b) and f has a maximum at (a,b) if $f_{xx}(a,b) < 0$ or $f_{yy}(a,b) < 0$ and f has a minimum at (a,b) if $f_{xx}(a,b) > 0$ or $f_{yy}(a,b) > 0$

(iii) If $H = 0$ then f may or may not have extreme value at (a,b)

Note: (a,b) is called a saddle point if it is a stationary point of f and has neither maximum nor minimum at (a,b).

Example 3 Examine the function $f(x,y) = 2x^2 - xy + 2y^2 - 20x$ for extreme values.

Solution: $f_x = 4x - y - 20$, $f_y = -x + 4y$

So, $f_x = 0$ and $f_y = 0 \Rightarrow 4x - y - 20 = 0$ and $-x + 4y = 0$. Solving we get $x = 16/3$ and $y = 4/3$. So $(16/3, 4/3)$ is the only stationary point. Also, $f_{xx} = 4$, $f_{xy} = -1$, $f_{yy} = 4$ and at $(16/3, 4/3)$

$$H = f_{xx}f_{yy} - f_{xy}^2 = 16 - 1 = 15 > 0 \quad \text{and} \quad f_{xx} > 0$$

So, f has a minimum at $(16/3, 4/3)$ and the minimum value is

$$= 2\times\frac{16}{3}\times\frac{16}{3} - \frac{16}{3}\times\frac{4}{3} + 2\times\frac{4}{3}\times\frac{4}{3} - 20\times\frac{16}{3}$$
$$= -\frac{160}{3}$$

Example 4 Examine $f(x,y) = x^3 + y^3 + 3xy$ for maximum and minimum values and for stationary points.

Solution: Here $f_x = 3(x^2+y)$, $f_y = 3(y^2+x)$

Now $f_x = 0$ and $f_y = 0 \Rightarrow x^2+y=0$ and $y^2+x=0$.

$\Rightarrow x=0, y=0$ and $x=-1, y=-1$. So, $(0,0)$ and $(-1,-1)$ are the two stationary points.

At $(0,0)$, $f_{xx} = 6\times 0 = 0$ and $f_{xy}=3$ and $f_{yy} = 6\times 0 = 0$

So, $H = f_{xx}f_{yy} - f_{xy}^2 = -9 < 0$ and hence f has no extreme value at $(0,0)$. i.e., $(0,0)$ is a saddle point.

At $(-1,-1)$, $f_{xx} = -6$, $f_{xy} = 3$, and $f_{yy} = -6$. So, we have $H = f_{xx}f_{yy} - f_{xy}^2 = 36-9 = 27 > 0$ and $f_{xx} < 0$. Hence f has maximum at $(-1,-1)$ and the maximum value is 1

Example 5 Examine the function $f(x,y) = 2(x-y)^2 - x^4 - y^4$ for extreme values.

Solution: $f_x = 4(x-y) - 4x^3$, $f_y = -4(x-y) - 4y^3$, $f_{xx} = 4-12x^2$, $f_{xy} = -4$ and $f_{yy} = 4-12y^2$. Now $f_x = 0$ and $f_y = 0 \Rightarrow$

$4(x-y) - 4x^3 = 0$ and $-4(x-y) - 4y^3 = 0 \Rightarrow x-y-x^3 = 0$ and $x-y+y^3 = 0$.

Solving, we get $(0,0)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ are the stationary points.

At $(\sqrt{2}, -\sqrt{2})$, $f_{xx} = -20$, $f_{xy} = -4$ and $f_{yy} = -20$ and hence

$H = f_{xx}f_{yy} - f_{xy}^2 = 400-16 = 384 > 0$

and $f_{xx} < 0$. So f has a maximum at $(\sqrt{2}, -\sqrt{2})$ and its value $= 16-8 = 8$

At $(-\sqrt{2}, \sqrt{2})$, $f_{xx} = -20$, $f_{xy} = -4$ and $f_{yy} = -20$

Hence $H = f_{xx}f_{yy} - f_{xy}^2 = 400-16 = 384 > 0$ and $f_{xx} < 0$

So, f has a maximum at $(-\sqrt{2}, \sqrt{2})$ and its value is $= 8$

At $(0,0)$, $f_{xx} = 4$, $f_{xy} = -4$ $f_{yy} = 4$. So, $H = f_{xx}f_{yy} - f_{xy}^2 = 0$

So, from this we can not conclude whether f has an extreme value $(0,0)$ or not. we need further consideration. We have $f(0,0) = 0$

For points in the neighbourhood of $(0,0)$ where $y=0$, we have $f(x,y) = x^2(2-x^2) > 0$

But for points in the neighbourhood of $(0,0)$ where $y=x$, $f(x,y) = -2x^4 < 0$

Thus $(0,0)$ is not an extreme point of f.