Example 6 We give an example to show that f attains an extremum at some point (a,b) where f_x and f_y do not exist.

Consider the function $f(x,y) = |x| + |y|$, $x,y \in \mathbb{R}$

Now $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{|h|}{h} = 1$ or -1

according as $h>0$ or $h<0$. Since $\lim_{h\to 0} \frac{|h|}{h}$ does not exist, so $f_x(0,0)$ does not exist. Similarly, $f_y(0,0)$ does not exist.

But $f(x,y) = |x| + |y| \geqslant f(0,0) = 0$ for every point (x,y) in any neighbourhood of $(0,0)$, confirming that f has a minimum at $(0,0)$.

Example 7 Show that the function f given by $f(x,y) = y^2 + 2x^2y + 2x^4$ has a minimum at $(0,0)$

Solution: $f(x,y) = y^2 + 2x^2y + 2x^4$

$f_x = 4xy + 8x^3$, $f_y = 2y + 2x^2$

Since $f_x(0,0) = 0$ and $f_y(0,0) = 0$, $(0,0)$ is a stationary point of f

$f_{xx} = 4y + 24x^2$ $f_{xy} = 4x$ $f_{yy} = 2$

Hence $f_{xx} f_{yy} - f_{xy}^2 = 0 \times 2 - 0 = 0$ at $(0,0)$

So, we can not conclude anything from this result.

Now $f(x,y) - f(0,0) = y^2 + 2x^2y + 2x^4 = y^2 + 2x^2y + x^4 + x^4$

$= (y+x^2)^2 + x^4 \geqslant 0$ in any neighbourhood of $(0,0)$. Hence, from the definition f has a minimum at $(0,0)$.

Exercise 1. Examine the following functions for extreme values:

(i) $2x^2 - xy + 2y^2 - 20x$

(ii) $x^2 + y^2 + (x+y+1)^2$

(iii) $y^2 + 4xy + 3x^2 + y^3$

Extreme value of functions of three variables:

Let $S \subset \mathbb{R}^3$ and $f: S \to \mathbb{R}$ be a function of three independent variables x, y, and z. Let (a,b,c) be an interior point of S. f is said to have an extreme value at (a,b,c) if, $\Delta f = f(x,y,z) - f(a,b,c)$ does not change sign in some neighbourhood of (a,b,c). When $\Delta f \geq 0$, f has a (local) minimum at (a,b,c) and when $\Delta f \leq 0$, f has a maximum at (a,b,c).

We state a necessary condition for extremum:

Theorem 3 Let $S \subset \mathbb{R}^3$ and $f: S \to \mathbb{R}$ be such that f_x, f_y and f_z exist at an interior point (a,b,c) of S. If f has an extreme value at (a,b,c), then

$$f_x(a,b,c) = 0, \quad f_y(a,b,c) = 0 \text{ and } f_z(a,b,c) = 0$$

Now we state a sufficient condition for existence of extremum:

Theorem 4 Let $S \subset \mathbb{R}^3$ and $f: S \to \mathbb{R}$ be such that f possesses continuous first and second order partial derivatives in a neighbourhood of an interior point (a,b,c) of S such that $f_x(a,b,c) = 0$, $f_y(a,b,c) = 0$, $f_z(a,b,c) = 0$. Then f has an extreme value at (a,b,c) if the quadratic form

$$d^2f(a,b,c) = f_{xx}(a,b,c)(dx)^2 + f_{yy}(a,b,c)(dy)^2 + f_{zz}(a,b,c)(dz)^2 + 2f_{xy}(a,b,c)\,dx\,dy$$
$$+ 2f_{yz}(a,b,c)\,dy\,dz + 2f_{zx}(a,b,c)\,dz\,dx$$

is definite, i.e., $d^2f(a,b,c)$ has same sign for arbitrary values of dx, dy and dz.

If $d^2f(a,b,c)$ is positive definite f has minimum value at (a,b,c) and if $d^2f(a,b,c)$ is negative definite f has maximum value at (a,b,c).

If $d^2f(a,b,c)$ is indefinite then (a,b,c) is a saddle point of f, i.e., f has no extreme value at (a,b,c). If $d^2f(a,b,c)$ is semi-definite then further investigation is necessary for drawing any conclusion regarding extreme value of f, if any, at (a,b,c).

Working rule to examine the ~~ext~~ existence of extreme values of a function of not more than three variables:

1. **Step 1** Find (a,b,c) where f_x, f_y, f_z vanish

Step 2 Find the principal minors of the matrix

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

(partial derivatives of f are computed at (a,b,c) corresponding to the quadratic form

$$d^2f(a,b,c) = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dx\,dz$$

(here $f_{xy} = f_{yx}$, $f_{yz} = f_{zy}$ and $f_{xz} = f_{zx}$))

i.e., $A = f_{xx}$, $B = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ and $C = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$

Step 3 If $A>0, B>0, C>0$, i.e., $d^2f(a,b,c)$ is positive definite then f has a minimum value at (a,b,c)

If $A<0$, $B>0$ and $C<0$, i.e., the principal minors (starting from the first order) are alternatively negative and positive, which occurs when $d^2f(a,b,c)$ is negative definite, then f has a maximum value at (a,b,c)

If $B<0$ (or in the other cases when) $d^2f(a,b,c)$ is indefinite then f has no extreme value at (a,b,c)

If $B=0$, then nothing can be concluded at this stage and further investigation is necessary.

Example 8 Examine for existence of maxima/minima of the function

$$f(x,y,z) = x^2 + y^2 + 3z^2 - xy + 2zx + yz$$

Solution : $f(x,y,z) = x^2+y^2+3z^2-xy+2zx+yz$

$f_x = 2x-y+2z$, $f_y = 2y-x+z$, $f_z = 6z+2x+y$

Since the coefficient matrix $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{bmatrix}$ is non-singular

as $\begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 3 & 2 & 5 \\ 4 & 1 & 8 \end{vmatrix}$ $R_1' \to R_1+2R_2$, $R_3' \to R_3+2R_2$

$= \begin{vmatrix} 3 & 5 \\ 4 & 8 \end{vmatrix} = 24-20 = 4 \neq 0$.

So, the system $f_x=0$, $f_y=0$ and $f_z=0$ has only one solution $(0,0,0)$. So, $(0,0,0)$ is the only stationary point of f.

Now $f_{xx}=2$, $f_{yy}=2$ $f_{zz}=6$, $f_{xy}=f_{yx}=-1$

$f_{xz}=f_{zx}=2$, $f_{yz}=f_{zy}=1$

So, $f_{xx}(0,0,0)=2$, $f_{yy}(0,0,0)=2$, $f_{zz}(0,0,0)=6$

$f_{xy}(0,0,0)=f_{yx}(0,0,0)=-1$, $f_{yz}(0,0,0)=f_{zy}(0,0,0)=1$

$f_{zx}(0,0,0)=f_{xz}(0,0,0)=2$

Hence at $(0,0,0)$ $A=f_{xx}>0$, $B=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1=3>0$

and $C=\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{vmatrix} = 4>0$

Hence $d^2f(0,0,0)$ is positive definite. Hence f has minimum value at $(0,0,0)$.