Exercise 1. Show that the following functions have minima at the indicated points:

(a) $x^4+y^4+z^4-4xyz$ at $(1,1,1)$

(b) $2xyz-4zx-2yz+x^2+y^2+z^2-2x-4y+4z$ at $(1,2,0)$

2.(A)

B. Application of Lagrange's Method of undetermined multipliers for determination of stationary points:

To find the extrema of the functions of three variables $f(x,y,z)$ ... (1)

Subject to the condition $\phi(x,y,z)=0$ ... (2)

The necessary condition for an extrema of f is

$$df = f_x\,dx + f_y\,dy + f_z\,dz = 0 \quad \ldots (3)$$

Moreover from (2), taking differentials

$$d\phi = \phi_x\,dx + \phi_y\,dy + \phi_z\,dz = 0 \quad \ldots (4)$$

Next we introduce one undetermined multiplier λ.

Multiply (4) by λ and add to (3).

$$(f_x+\lambda\phi_x)dx + (f_y+\lambda\phi_y)dy + (f_z+\lambda\phi_z)dz = 0 \quad \ldots (5)$$

Choose λ such that $f_z+\lambda\phi_z = 0$ ... (6)

Then (5) becomes

$$(f_x+\lambda\phi_x)dx + (f_y+\lambda\phi_y)dy = 0 \quad \ldots (7)$$

Since dx and dy are independent increments,

$$f_x+\lambda\phi_x = 0 \text{ and } f_y+\lambda\phi_y = 0 \quad \ldots (8)$$

Solving (2), (6) and (8), obtain x, y, z and λ. Using (1), we get the stationary points.

<u>A second Approach</u>: For practical purposes, the process of obtaining equations (6) and (8) of the above section (A) may be put in the following form:

we form the Lagrangian function $F = f + \lambda\phi$. At a stationary point of F, $dF = 0$, the necessary condition for which is $F_x = 0, F_y = 0, F_z = 0$ which are same as (6) and (8)

(B) A stationary point will be an extreme point of f if d^2F be definite, i.e., if d^2F keeps the same sign and will be a maximum if $d^2F < 0$ and minimum if $d^2F > 0$ i.e., maximum when d^2F is negative definite and minimum when d^2F is positive definite.

Note: In general, Lagrangian Function F is denoted by L.

Example 1 : Suppose x, y are connected by the relation $x^2+y^2=1$, find the extreme values of $f(x,y) = 7x^2 + 8xy + y^2$

Solution : The Lagrangian function is

$$L(x,y) = 7x^2+8xy+y^2 + \lambda(x^2+y^2-1)$$, λ is a Lagrangian multiplier.

For, stationary points, $L_x = 0, L_y = 0$

So, $2(7+\lambda)x + 8y = 0$

and $8x + 2(1+\lambda)y = 0$

Since x, y are not both zero as $x^2+y^2 = 1$, so

$$\begin{vmatrix} 2(7+\lambda) & 8 \\ 8 & 2(1+\lambda) \end{vmatrix} = 0 \text{ or, } (7+\lambda)(1+\lambda) - 16 = 0$$

or, $\lambda^2 + 8\lambda - 9 = 0 \Rightarrow (\lambda+9)(\lambda-1) = 0 \Rightarrow \lambda = -9, 1$

For $\lambda = -9$, we have $x - 2y = 0$. Since $x^2+y^2 = 1$,

we have $x = \pm\frac{2}{\sqrt{5}}$, $y = \pm\frac{1}{\sqrt{5}}$

For $\lambda = 1$, $2x + y = 0$. Since $x^2+y^2=1$, we have $x = \pm\frac{1}{\sqrt{5}}$, $y = \mp\frac{2}{\sqrt{5}}$

Thus we have four stationary points $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(-\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$,

$\left(\frac{1}{\sqrt5}, -\frac{2}{\sqrt5}\right)$ and $\left(-\frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right)$

For $\lambda = -9$, $L_x = -4x + 8y$, $L_y = 8x - 16y$,

$dL = L_x\,dx + L_y\,dy$

$$d^2L = L_{xx}(dx)^2 + 2L_{xy}\,dx\,dy + L_{yy}(dy)^2$$

$$= -4(dx)^2 + 16\,dx\,dy - 16(dy)^2$$

Since $x^2+y^2=1$, we have $x\,dx + y\,dy = 0$

Eliminating dy,

$$d^2L = -4(dx)^2 + 16\left(dx + \frac{x}{y}dx\right)\left(-\frac{x}{y}dx\right)$$

$$= -\left[4 + 16\frac{x}{y} + 16\frac{x^2}{y^2}\right](dx)^2$$

At $\left(\frac{2}{\sqrt5}, \frac{1}{\sqrt5}\right)$, $d^2L = -\left[4+32+64\right](dx)^2 < 0$

Similarly at $\left(-\frac{2}{\sqrt5}, -\frac{1}{\sqrt5}\right)$, $d^2L < 0$

So, d^2L is negative definite at $\left(\pm\frac{2}{\sqrt5}, \pm\frac{1}{\sqrt5}\right)$

Hence f has maximum value 9 at $\left(\pm\frac{2}{\sqrt5}, \pm\frac{1}{\sqrt5}\right)$

For $\lambda = 1$, $L_x = 16x + 8y$, $L_y = 8x + 4y$

Since $dL = L_x\,dx + L_y\,dy$

$$d^2L = L_{xx}(dx)^2 + 2L_{xy}\,dx\,dy + L_{yy}(dy)^2$$

$$= 16(dx)^2 + 16\,dx\,dy + 4(dy)^2$$

As $x^2+y^2=1$ gives $x\,dx + y\,dy = 0$. Eliminating dy,

we have $d^2L = 16(dx)^2 - 16\frac{x}{y}(dx)^2 + 4\frac{x^2}{y^2}(dx)^2$

$$= 4\left(4 - 4\frac{x}{y} + \frac{x^2}{y^2}\right)(dx)^2$$

At $\left(\frac{1}{\sqrt5}, \frac{-2}{\sqrt5}\right)$, $d^2L = 4\left(4+2+\frac14\right)(dx)^2 > 0$

At $\left(\frac{-1}{\sqrt5}, \frac{2}{\sqrt5}\right)$, $d^2L = 4\left(4+2+\frac14\right)(dx)^2 > 0$

So, d^2L is positive definite at $\left(\pm\frac{1}{\sqrt5}, \mp\frac{2}{\sqrt5}\right)$

Hence f has minimum value -1 at $\left(\pm\frac{1}{\sqrt5}, \mp\frac{2}{\sqrt5}\right)$

Example 2 If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, show that a stationary value of $a^3x^2+b^3y^2+c^3z^2$ is given by $ax=by=cz$ and this gives an extreme value if $abc(a+b+c)$ is positive.

Solution: Let $f(x,y,z)=a^3x^2+b^3y^2+c^3z^2$ where (x,y,z) satisfies $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1=0$

The Lagrangian function is

$L(x,y,z)=a^3x^2+b^3y^2+c^3z^2+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$ where constant λ is Lagrangian multiplier.

For stationary values, $dL=0$

$\Rightarrow L_x = 2a^3x-\frac{\lambda}{x^2}=0,\ L_y=2b^3y-\frac{\lambda}{y^2}=0,\ L_z=2c^3z-\frac{\lambda}{z^3}=0$

$\Rightarrow a^3x^3=b^3y^3=c^3z^3=\frac{\lambda}{2}$ ... (1)

$\Rightarrow ax=by=cz$

$\Rightarrow \frac{a}{\frac{1}{x}}=\frac{b}{\frac{1}{y}}=\frac{c}{\frac{1}{z}}=a+b+c$ (since $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$)

$\Rightarrow x=\frac{a+b+c}{a},\ y=\frac{a+b+c}{b},\ z=\frac{a+b+c}{c}$

Thus the stationary point is $\left(\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}\right)$

$L_{xx}=2a^3+\frac{2\lambda}{x^3},\ L_{yy}=2b^3+\frac{2\lambda}{y^3},\ L_{zz}=2c^3+\frac{2\lambda}{z^3}$

$L_{xy}=0,\ L_{yz}=0,\ L_{zx}=0$

Then $d^2L = L_{xx}(dx)^2+L_{yy}(dy)^2+L_{zz}(dz)^2$

$= 6\left[a^3(dx)^2+b^3(dy)^2+c^3(dz)^2\right]$ [by (1)]

Since $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ $\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}=0$

$\Rightarrow dz = -\left(\frac{z^2}{x^2}dx+\frac{z^2}{y^2}dy\right) = -\left(\frac{a^2}{c^2}dx+\frac{b^2}{c^2}dy\right)$ at the stationary point.

Thus $d^2L = 6\left[a^3(dx)^2+b^3(dy)^2+c^3\frac{(a^2dx+b^2dy)^2}{c^4}\right]$

$= \frac{6}{c}\left[a^3(c+a)(dx)^2+b^3(c+b)(dy)^2+2a^2b^2dx\,dy\right]$

d^2L is definite if $\begin{vmatrix} \frac{6a^3}{c}(c+a) & \frac{6a^2b^2}{c} \\ \frac{6a^2b^2}{c} & \frac{6b^3}{c}(c+b)\end{vmatrix} > 0$

i.e., if $\frac{a^3b^3}{c^2}\left[(b+c)(c+a)-ab\right]>0$

i.e., if $\frac{a^2b^2}{c^2}abc(a+b+c)>0$

i.e., if $abc(a+b+c)>0$

So, stationary values of $a^3x^2+b^3y^2+c^3z^2$ is given by $ax=by=cz(=a+b+c)$ and it has extreme value at the stationary point if $abc(a+b+c)>0$

Example 3 Prove that the volume of the greatest parallelopiped That can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{8abc}{3\sqrt{3}}$.

Solution: Here the problem is to find the maximum value of $f(x,y,z)=(2x)(2y)(2z)=8xyz$ where x,y,z are all positive satisfying $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. $\quad\cdots(1)$

Here the Lagrangian function is $L=8xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$ where λ is a Lagrangian undetermined multiplier.

For stationary points, $dF=0$ where

$$dF = \left(8yz + \frac{2\lambda x}{a^2}\right)dx + \left(8zx + \frac{2\lambda y}{b^2}\right)dy + \left(8xy + \frac{2z}{c^2}\right)dz$$

So, $L_x = 8yz + \frac{2\lambda x}{a^2} = 0$, $L_y = 8zx + \frac{2\lambda y}{b^2} = 0$, $L_z = 8xy + \frac{2\lambda z}{c^2} = 0$ $\quad \cdots (2)$

Now $xL_x + yL_y + zL_z = 0$ gives

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

or, $\lambda = -12xyz \quad \cdots (3)$

From (1) and (3) $x = \frac{a}{\sqrt{3}}$, $y = \frac{b}{\sqrt{3}}$, $z = \frac{c}{\sqrt{3}}$, $\lambda = -\frac{4abc}{\sqrt{3}}$

So, Here the stationary point is $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$

Again $d^2L = 2\lambda\left(\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \frac{(dz)^2}{c^2}\right) + 16z\,dx\,dy + 16x\,dy\,dz + 16y\,dz\,dx$

So, $d^2L = -\frac{8abc}{\sqrt{3}}\left(\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \frac{(dz)^2}{c^2}\right) + \frac{16}{\sqrt{3}}\left(c\,dx\,dy + a\,dy\,dz + b\,dz\,dx\right)$ at $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ $\quad \cdots (4)$

Also from (1) $\frac{x\,dx}{a^2} + \frac{y\,dy}{b^2} + \frac{z\,dz}{c^2} = 0$

or, $\frac{dx}{a} + \frac{dy}{b} + \frac{dz}{c} = 0$ at $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$

Squaring, we get

$$\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \frac{(dz)^2}{c^2} + 2\frac{dx\,dy}{ab} + 2\frac{dy\,dz}{bc} + \frac{2\,dz\,dx}{ca} = 0$$

Multiplying by abc, we have,

$$abc\left(\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \frac{(dz)^2}{c^2}\right) + 2c\,dx\,dy + 2a\,dy\,dz + 2b\,dz\,dx = 0$$

or, $\frac{8abc}{\sqrt{3}}\left(\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \frac{(dz)^2}{c^2}\right) + \frac{16}{\sqrt{3}}\left[c\,dx\,dy + a\,dy\,dz + b\,dz\,dx\right] = 0$

or, $\frac{16}{\sqrt{3}}\left(c\,dx\,dy + a\,dy\,dz + b\,dz\,dx\right) = -\frac{8abc}{\sqrt{3}}\left(\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \frac{(dz)^2}{c^2}\right)$

So, $d^2L = -\frac{16abc}{\sqrt{3}}\left(\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \frac{(dz)^2}{c^2}\right) < 0$. So $f(x,y,z) = 8xyz$ has a maximum value at $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ and the maximum value $= \frac{8abc}{3\sqrt{3}}$

So, the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{8abc}{3\sqrt{3}}$.