Example: $\mathbb{R}^2$ has dimension 2, $\mathbb{R}^3$ has dimension 3, $\mathbb{R}^n$ has dimension n.

Real Quadratic form

An expression of the form $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} x_i x_j$ where a_{ij} are real numbers and $a_{ij} = a_{ji}$, is said to be a real quadratic form in n variables $x_1, x_2, \dots, x_n$.

The matrix notation of the quadratic form is $X^t A X$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $A = (a_{ij})_{n\times n}$. A is a real symmetric matrix since $a_{ij} \in \mathbb{R}$ and $a_{ij} = a_{ji}$.

To every real quadratic form in n variables, an $n\times n$ real symmetric matrix is associated, which is said to be the matrix of the quadratic form.

Examples $5x_1^2 + 2x_1x_2 - x_2^2$ is a real quadratic form in two variables x_1, x_2. The matrix form is

$X^t A X$ where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $A = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}$

So, The associated matrix is $\begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}$

2. $x_1x_2 - x_2x_3$ is a real quadratic form in three variables x_1, x_2, x_3.

The associated matrix is $\begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}$

3. $x_1^2 - x_2^2 + 2x_3^2 + 3x_1x_2 + 5x_2x_3 + 6x_3x_1$ is a real quadratic form in three variables

The associated matrix is $\begin{pmatrix} 1 & 3/2 & 3 \\ 3/2 & -1 & 5/2 \\ 3 & 5/2 & 2 \end{pmatrix}$

[To find the associated matrix, we see the order the matrix is the no. of variables. Now the coefficients of x_1^2, x_2^2 and x_3^2 gives the elements a_{11}, a_{22} and a_{33}. a_{12} and a_{21} are same and both are half of the coefficient of $x_1 x_2$. Similarly, we get a_{13} and a_{31} and a_{23} and a_{32}]

A real Quadratic form $Q = X^t A X$ takes the value 0 when $x_1 = x_2 = \cdots = x_n = 0$ i.e., $X = 0$, But Q takes different values for different non-zero ~~values~~ of X.

A real quadratic form $Q = X^t A X$ is said to be

(i) positive definite if $Q > 0$ for all $X \neq 0$

(ii) positive semi definite if $Q \geq 0$ for all X and $Q = 0$ for some $X \neq 0$

(iii) negative definite if $Q < 0$ for all $X \neq 0$

(iv) negative semi definite if $Q \leq 0$ for all X and $Q = 0$ for some $X \neq 0$

(v) indefinite if $Q \geq 0$ for some $X \neq 0$ and $Q \leq 0$ for some other $X \neq 0$

Result 1. If A be a real $n \times n$ symmetric matrix of rank r then it is congruent to a diagonal matrix $D = \begin{pmatrix} I_m & & \\ & -I_{r-m} & \\ & & 0 \end{pmatrix}$

[Congruent operation means one row operation and the same column operation]

Example 1:

Consider the symmetric matrix $A = \begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ -5 & -2 & 10 \end{pmatrix}$

We now apply congruent operation on A

$$A \xrightarrow{R_3+R_1} \begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix} \xrightarrow{C_3+C_1} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$

$$\xrightarrow{R_3+2R_2} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_3+2C_2} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{\sqrt{5}}R_1} \begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{1}{\sqrt{5}}C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

So, A is congruent to I_3. Hence rank of $A = 3$

Example 2: Consider the matrix $A = \begin{pmatrix} 3 & 4 & 2 \\ 4 & -1 & -3 \\ 2 & -3 & 0 \end{pmatrix}$

We now apply congruent operation on A

$$A \xrightarrow{R_2-\frac{4}{3}R_1} \begin{pmatrix} 3 & 4 & 2 \\ 0 & -\frac{19}{3} & -\frac{17}{3} \\ 2 & -3 & 0 \end{pmatrix} \xrightarrow{C_2-\frac{4}{3}C_1} \begin{pmatrix} 3 & 0 & 2 \\ 0 & -\frac{19}{3} & -\frac{17}{3} \\ 2 & -\frac{17}{3} & 0 \end{pmatrix}$$

$$\xrightarrow{R_3-\frac{2}{3}R_1} \begin{pmatrix} 3 & 0 & 2 \\ 0 & -\frac{19}{3} & -\frac{17}{3} \\ 0 & -\frac{17}{3} & -\frac{4}{3} \end{pmatrix} \xrightarrow{C_3-\frac{2}{3}C_1} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -\frac{19}{3} & -\frac{17}{3} \\ 0 & -\frac{17}{3} & -4/3 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{\sqrt{3}}R_1} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\frac{19}{3} & -\frac{17}{3} \\ 0 & -\frac{17}{3} & -4/3 \end{pmatrix} \xrightarrow{\frac{1}{\sqrt{3}}C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{19}{3} & -\frac{17}{3} \\ 0 & -\frac{17}{3} & -4/3 \end{pmatrix}$$

$$\xrightarrow{R_3 - \frac{17}{19}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{19}{3} & -\frac{17}{3} \\ 0 & 0 & \frac{71}{19} \end{pmatrix} \xrightarrow{C_3 - \frac{17}{19}C_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{19}{3} & 0 \\ 0 & 0 & \frac{71}{19} \end{pmatrix}$$

$$\xrightarrow{\sqrt{\frac{19}{3}}\, R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{\frac{19}{3}} & 0 \\ 0 & 0 & \frac{71}{19} \end{pmatrix} \xrightarrow{\sqrt{\frac{19}{3}}\, C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{71}{19} \end{pmatrix}$$

$$\xrightarrow{\sqrt{\frac{71}{19}}\, R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sqrt{\frac{71}{19}} \end{pmatrix} \xrightarrow{\sqrt{\frac{71}{19}}\, C_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

~~So A is congruent to~~ $\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

$$\xrightarrow{C_2 \leftrightarrow C_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So, A is congruent to the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Here rank of $A = 3$

If the quadratic form be $Q = X^t A X$

and A is congruent to $\begin{pmatrix} I_m & & \\ & -I_{r-m} & \\ & & O \end{pmatrix} = D$

then D has m positive 1 and r-m −1

The rank of D and hence of A is r. m and r are invariant i.e we always get same values irrespective of the congruent operations.

Then. Q in (i) positive definite if $r = n$ and, $m = r$

(ii) positive semi-definite, if $r < n$, $m = r$

(iii) negative definite if $r = n$, $m = 0$

(iv) negative semi definite if $r < n$, $m = 0$

(v) indefinite if $r \leq n$, $0 < m < r$

(r is called rank of the quadratic form, m = index and $m-(r-m) = 2m-r$ = signature)

By suitable transformation, a Quadratic form ~~Can be tra~~ $Q = X^t A X$ can be ~~transformed~~ transformed to $X^t D X$ where $D = \begin{pmatrix} I_m & \\ & -I_{r-m} \\ & & 0 \end{pmatrix}$

$X^t D X$ is called the normal form.
(sometime D is called also normal form)

~~Some problem~~

Some Problems: 1. Reduce the quadratic form $5x^2 + y^2 + 10z^2 - 4yz - 10zx$ in three variables to the normal form and show that it is positive definite.

Solution: The associated symmetric matrix is $A = \begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ -5 & -2 & 10 \end{pmatrix}$

Let us apply congruence operation on A to reduce it to the normal form

$$A \xrightarrow{R_3+R_1} \begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix} \xrightarrow{C_3+C_1} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$

$$\xrightarrow{R_3+2R_2} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_3+2C_2} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{\sqrt5}R_1} \begin{pmatrix} \sqrt5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{1}{\sqrt5}C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So, the normal form is $x^2+y^2+z^2$

The rank of the quadratic form $r = 3$ and $m =$ index $= 3$. So, the quadratic form is positive definite. Here the signature $= 2m - r = 6 - 3 = 3$

2. Show that the quadratic form

$x^2+2y^2+3z^2-2xy+4yz$ in three variables is indefinite

Solution: The associated matrix is $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$

Let us apply congruence operations on A.

$$A \xrightarrow{R_2+R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix} \xrightarrow{C_2+C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3-2R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{C_3-2C_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The normal form is $x^2+y^2-z^2$

Here rank $r = 3$ and $m = 2$

So, the quadratic form is indefinite.

Exercises : 1. Reduce the following quadratic forms in three variables to their normal form. Find the rank, index and signature

(i) $2x^2+5y^2+10z^2+4xy+12yz+6xz$

(ii) $x^2+2y^2+6xz+4yz$

(iii) $xy+yz+zx$