2. Reduce the following quadratic forms in three variables to the normal and find the nature of them

(i) $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 4x_2x_3 + 2x_1x_3$

(ii) $x^2 + 2y^2 + 2z^2 + 2xy + 2xz$

(iii) $x^2 + 2y^2 + z^2 + 4xy$

(iv) $5x^2 + y^2 + 5z^2 + 4xy - 8xz - 4yz$

For a square matrix $A = [a_{ij}]_{n\times n}$ of order n, we define $\det(A - xI_n) = 0$ as the characteristic equation of the square matrix A of order n. (I_n is the identity matrix of order n)

Let $A = [a_{ij}]_{3\times 3}$ Then $\det(A - xI_3) = \begin{vmatrix} a_{11}-x & a_{12} & a_{13} \\ a_{21} & a_{22}-x & a_{23} \\ a_{31} & a_{32} & a_{33}-x \end{vmatrix}$

So, the characteristic equation of A is

$$\begin{vmatrix} a_{11}-x & a_{12} & a_{13} \\ a_{21} & a_{22}-x & a_{23} \\ a_{31} & a_{32} & a_{33}-x \end{vmatrix} = 0$$

A root of the characteristic equation of a square matrix A is said to be an eigen value of A.

Examples 1. Let $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

The characteristic equation of A is $\begin{vmatrix} -x & -1 \\ -1 & -x \end{vmatrix} = 0$

Or, $x^2 - 1 = 0$, $x = \pm 1$. So, the eigenvalues are $1, -1$

2. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

The characteristic equation of A is $\begin{vmatrix} -x & -1 \\ 1 & -x \end{vmatrix} = 0$

Then $x^2+1=0$, $x=\pm i$. So, the eigenvalues are $i, -i$

3. Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}$. The characteristic equation

of A is $\begin{vmatrix} 1-x & -1 & 0 \\ 1 & 2-x & -1 \\ 3 & 2 & -2-x \end{vmatrix} = 0$

or, $(1-x)\begin{vmatrix} 2-x & -1 \\ 2 & -2-x \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & -2-x \end{vmatrix} = 0$

or, $(1-x)(x^2-2) + (1-x) = 0$

or, $(1-x)(x^2-1) = 0$ or, $x = 1, 1, -1$

So, the eigenvalues of A are $1, 1, -1$.

Let A be a real square matrix of order n. Then a non-null vector $X \in \mathbb{R}^n$ is said to be an eigenvector of A if there exists real numbers λ such that

$$AX = \lambda X$$

So, If X be an eigenvector then X is non-null

and $AX = \lambda X$ for some real number λ.

or, $(A - \lambda I_n)X = O$.

This is a homogeneous system of equations in n unknowns. Since it has a non-null solution, $\det(A-\lambda I_n) = 0$

So, this implies λ is an eigenvalue of A. So, for each eigenvector there is an eigenvalue.

Also if λ be an eigenvalue of A. $\det(A-\lambda I_n)=0$

So, the homogeneous system $(A-\lambda I_n)X = O$ has a non-null solution. So, corresponding to each eigenvalue there is an eigenvector.

Problems for determination of Eigen values and Eigen vectors :

1. Find the eigenvalues of $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$ and their corresponding eigenvectors.

Solution: The characteristic equation of A is

$$\begin{vmatrix} 1-x & 3 \\ 4 & 5-x \end{vmatrix} = 0$$

or, $(1-x)(5-x) - 12 = 0$

or, $x^2 - 6x + 5 - 12 = 0$ or, $x^2 - 6x - 7 = 0$

or, $x^2 - 7x + x - 7 = 0$ or, $x(x-7) + (x-7) = 0$

or, $(x-7)(x+1) = 0$ or, $x = -1, 7$. So, the ~~Eigen~~ eigen values are $-1, 7$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to -1

Then $AX = -X$ or, $(A + I_2)X = \theta$

or, $2x_1 + 3x_1 = 0$

and $4x_1 + 6x_2 = 0$

The equivalent system is $x_1 + \frac{3}{2}x_2 = 0$ or, $x_1 = -\frac{3}{2}x_2$

So, the solution of the given system is $k\left(-\frac{3}{2}, 1\right)$, where $k \in \mathbb{R}$

So, the eigen vectors are $k\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$ or, equivalently, $c\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, where $c \in \mathbb{R}$ and $c \neq 0$

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigenvector corresponding to 7.

Then $AX = 7X$ and this gives

$$-6x_1 + 3x_2 = 0$$

$$4x_1 - 2x_2 = 0$$

The system is equivalent to $x_1 - \frac{1}{2}x_2 = 0$ or, $2x_1 = x_2$

Putting $x_1 = k$, $x_2 = 2k$, k is a real number

So, the eigenvectors are $k\begin{pmatrix}1\\2\end{pmatrix}$ where k is a non-zero real number.

2. Find the eigenvalues and the corresponding eigen vectors of the matrix $\begin{pmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 5\end{pmatrix}$

Solution: Let $A = \begin{pmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 5\end{pmatrix}$. Then the characteristic equation of A is
$$\begin{vmatrix}2-\lambda & 0 & 0\\ 0 & 3-\lambda & 0\\ 0 & 0 & 5-\lambda\end{vmatrix} = 0$$

or, $(2-\lambda)(3-\lambda)(5-\lambda) = 0$ or, $\lambda = 2, 3, 5$

So, the eigenvalues are 2, 3, 5.

Let $X = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ be an eigenvector corresponding to 2.

Then $AX = 2X$ and this gives the system as

$$x_2 = 0$$
$$3x_3 = 0$$

So, the solution of the system is $k\begin{pmatrix}1\\0\\0\end{pmatrix}$ where $k \in \mathbb{R}$

So, the eigenvectors are $k\begin{pmatrix}1\\0\\0\end{pmatrix}$, k is a non-zero real number.

Let $X = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ be the eigenvector corresponding to 3.

Then $AX = 3X$ and this gives the system as

$$-x_1 = 0$$
$$2x_3 = 0$$

So, the eigen vectors are $k\begin{pmatrix}0\\1\\0\end{pmatrix}$, $k \neq 0$ and $k \in \mathbb{R}$.

2 Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigenvector corresponding to 5.

So, $AX = 5X$. This gives the system as

$$-3x_1 = 0$$
$$-2x_2 = 0$$

So, the eigenvectors are $k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $k \neq 0$ and $k \in \mathbb{R}$.

3. Find all the eigenvalues and the corresponding eigenvector of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

The characteristic equation of A is $\begin{vmatrix} 1-x & 1 & 1 \\ -1 & -1-x & -1 \\ 0 & 0 & 1-x \end{vmatrix} = 0$

or, $(1-x) \begin{vmatrix} 1-x & 1 \\ -1 & -1-x \end{vmatrix} = 0$

or, $(1-x)(x^2 - 1 + 1) = 0$ or, $x^2(1-x) = 0$.

So, the eigenvalues are 0, 0, 1

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigenvector corresponding to 0.

Then $AX = 0$ or,

$$x_1 + x_2 + x_3 = 0$$
$$-x_1 - x_2 - x_3 = 0$$
$$x_3 = 0$$

or, The equivalent system is

$$x_1 + x_2 + x_3 = 0$$
$$x_3 = 0$$

Let $x_2 = k$ Then $x_1 = -k$ where k is a real number.

So, $k \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ are the eigen vectors of 0 where $k \in \mathbb{R}$ and $k \neq 0$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigenvector corresponding to 1

So, $AX = X$. This gives

$$x_2 + x_3 = 0$$
$$-x_1 - 2x_2 - x_3 = 0$$

So, let $x_3 = k$ then $x_2 = -k$

and $x_1 = k$

So, $k \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ are the eigenvectors of 1

where $k \in \mathbb{R}$ and $k \neq 0$

Now we state a theorem without proof. This theorem is called Cayley-Hamilton theorem.

Theorem 7 (Cayley-Hamilton theorem) Every square matrix satisfies its own characteristic equation.

Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$. Then the characteristic equation of A

is $x^2 - 6x - 7 = 0$ ——————— (1)

Now, Cayley-Hamilton Theorem says that A satisfies (1)

i.e., $A^2 - 6A - 7I_2 = 0$. (We can easily verify it here)

Where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Examples: 1. ~~Now~~ Verify Cayley-Hamilton for the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$. Express A^{-1} as polynomial of A and then compute A^{-1}.

Solution: The characteristic equation of A is $\begin{vmatrix} 1-x & 0 & 0 \\ 1 & 2-x & 1 \\ 2 & 3 & 2-x \end{vmatrix} = 0$

or, $(1-x)\left[(2-x)^2 - 3\right] = 0$ or, $(1-x)\left[x^2 - 4x + 1\right] = 0$, or, $-x^3 + 4x^2 - x + x^2 - 4x + 1 = 0$

or, $x^3 - 5x^2 + 5x - 1 = 0$. Now $A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 7 & 4 \\ 9 & 12 & 7 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & 0 & 0 \\ 20 & 26 & 15 \\ 35 & 45 & 26 \end{pmatrix}$

So, $A^3 - 5A^2 + 5A - I_3 = O$. So, Cayley Hamilton Theorem is satisfied.

Now $A^3 - 5A^2 + 5A = I_3$. If A^{-1} exists then $A^{-1} = A^2 - 5A + 5I_3$ as

$A(A^2 - 5A + 5I) = I_3$ So, $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$.

END