2. Real vector space $\mathbb{R}_{m\times n}$. Let V be the set of all $m\times n$ matrices over $\mathbb{R}$. Let us consider the matrix addition and scalar multiplication defined as follows. Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$. Then $A+B = (c_{ij})_{m\times n}$ where $c_{ij} = a_{ij} + b_{ij}$. Let $c \in \mathbb{R}$ and $A = [a_{ij}]_{m\times n}$ then the scalar multiplication

$cA = [d_{ij}]_{m\times n}$ where $d_{ij} = ca_{ij}$

With respect to these two operations, V1 – V10 are satisfied by V. Then V is a real vector space and is denoted by $\mathbb{R}_{m\times n}$

Theorem 1 In a vector space V over a field F,

(i) $0\alpha = \theta$ for all $\alpha \in V$

(ii) $c\theta = \theta$ for all $c \in F$

(iii) $(-1)\alpha = -\alpha$ for all $\alpha \in V$, 1 being the multiplicative identity in F.

(iv) $c\alpha = \theta$ implies either $c = 0$ or $\alpha = \theta$

Proof: (i) 0 is the zero element in F. So,

$0+0 = 0$ in F

$\Rightarrow (0+0)\alpha = 0\alpha$ in V

$\Rightarrow 0\alpha + 0\alpha = 0\alpha$ by V_9.

$-0\alpha \in V$, since $0\alpha \in V$

So, $-0\alpha + (0\alpha + 0\alpha) = -0\alpha + 0\alpha$

or, $(-0\alpha + 0\alpha) + 0\alpha = \theta$ by V_3 and V_5

or, $\theta + 0\alpha = \theta$, by V_5

or, $0\alpha = \theta$. by V_4

(ii) θ is the zero element in V. So,

$\theta + \theta = \theta$ in V

$\Rightarrow c(\theta+\theta) = c\theta$

$\Rightarrow c\theta + c\theta = c\theta$, by V_8

$-c\theta \in V$ as $c\theta \in V$. So,

$-c\theta + (c\theta + c\theta) = -c\theta + c\theta$

or, $(-c\theta + c\theta) + c\theta = \theta$ by V_3 and V_5

or, $\theta + c\theta = \theta$ by V_5

or, $c\theta = \theta$ by V_4

(iii) We have $\theta = 0\alpha$ by (i)

$= [1 + (-1)]\alpha$

$= 1\alpha + (-1)\alpha$ by V_9

$= \alpha + (-1)\alpha$ by V_{10}

So, $-\alpha + \theta = -\alpha + (\alpha + (-1)\alpha)$

$= (-\alpha + \alpha) + (-1)\alpha$ by V_3

$= \theta + (-1)\alpha$ by V_5

$= (-1)\alpha$ by V_4

or, $-\alpha = (-1)\alpha$ by V_4

or, $(-1)\alpha = -\alpha$

(iv) Let $c\alpha = \theta$ and let $c \neq 0$. Then c^{-1} exists in F.

Now $c\alpha = \theta \Rightarrow c^{-1}(c\alpha) = c^{-1}\theta$

$\Rightarrow (c^{-1}c)\alpha = c^{-1}\theta$ by V_7

$\Rightarrow 1\alpha = \theta$ by (ii)

$\Rightarrow \alpha = \theta$ by V_{10}

So, $c\alpha = \theta$ and $c \neq 0 \Rightarrow \alpha = \theta$

Contrapositively, $c\alpha = \theta$ and $\alpha \neq \theta \Rightarrow c = 0$

Hence $c\alpha = \theta$ ~~or~~ $\Rightarrow$ either $c = 0$ or $\alpha = \theta$

Definition: Let V be a vector space over a field. Let $\alpha_1, \alpha_2, \ldots, \alpha_r \in V$. A vector β is said to be a linear combination of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_r$ if β can be expressed as $\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r$ for some scalars $c_1, c_2, \ldots, c_r \in F$.

Example: 1. Let V be a real vector space and $\alpha, \beta, \gamma \in V$. Then $\alpha + \beta + \gamma$, $2\alpha + 3\beta - \gamma$, $\alpha + 0.\beta + 0.\gamma$ are all linear combinations of α, β and γ.

2. Consider the vector space $\mathbb{R}^n$. Then $(2,1,3)$ is a linear combination of $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ as

$$(2,1,3) = 2(1,0,0) + (0,1,0) + 3(0,0,1)$$

$(4,-2,6)$ is a linear combination of $(2,1,5)$, $(1,2,3)$, $(4,1,7)$

as $(4,-2,6) = (2,1,5) - 2(1,2,3) + (4,1,7)$

Linear dependence and Linear independence

A finite set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of a vector space V over a field F is said to be linearly dependent in V if there exist scalars $c_1, c_2, \ldots, c_n$, not all zero, in F such that

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = \theta \quad \cdots (i)$$

The set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is said to be linearly independent in V if the equality (i) is satisfied only when $c_1 = c_2 = \cdots = c_n = 0$.

i.e., if $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = \theta$ then $c_1 = c_2 = \cdots = c_n = 0$.

Examples 1. Examine if the set of vectors $\{(1,2,3), (3,6,9)\}$ is linearly dependent in $\mathbb{R}^3$

Solution: Here, let $\alpha = (1,2,3)$ and $\beta = (3,6,9)$

then $3\alpha - \beta = (0,0,0)$

So, $\{\alpha, \beta\}$ is linearly dependent.

2. Examine if the set of vectors $\{(2,1,1), (1,2,2), (1,1,1)\}$ is linearly dependent in $\mathbb{R}^3$

Solution: Let $\alpha = (2,1,1)$, $\beta = (1,2,2)$ and $\gamma = (1,1,1)$

Let us consider the relation $c_1\alpha + c_2\beta + c_3\gamma = (0,0,0)$, where $c_1, c_2, c_3 \in \mathbb{R}$

Then $(2c_1, c_1, c_1) + (c_2, 2c_2, 2c_2) + (c_3, c_3, c_3) = (0,0,0)$

$\Rightarrow 2c_1 + c_2 + c_3 = 0$, $c_1 + 2c_2 + c_3 = 0$, $c_1 + 2c_2 + c_3 = 0$

or, equivalently $2c_1 + c_2 + c_3 = 0$ ~~and~~ $\cancel{c_1 + 2c_2 + c_3 = 0}$
and $c_1 + 2c_2 + c_3 = 0$

So, $\dfrac{c_1}{1-2} = \dfrac{c_2}{1-2} = \dfrac{c_3}{4-1} = k$ (say) or, $c_1 = -k$, $c_2 = -k$, $c_3 = 3k$ where k is a real number.

Since k is arbitrary, there exist c_1, c_2, c_3 not all zero such that $c_1\alpha + c_2\beta + c_3\gamma = (0,0,0)$

For example $\alpha + \beta - 3\gamma = (0,0,0)$

So, the set of vector is linearly dependent.

3. Prove that the set of vectors $\{(1,3), (3,5)\}$ is linearly independent.

Let $c_1(1,3) + c_2(3,5) = (0,0)$

$\Rightarrow \quad c_1 + 3c_2 = 0$

and $3c_1 + 5c_2 = 0$

This is a homogenious system of equation $Ax = 0$

Where $A = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$, $x = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Here $\det A = \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = 5 - 9 = -4 \neq 0$. So, A^{-1} exists

So, $A^{-1}(Ax) = A^{-1}0$

$\Rightarrow Ix = 0$ $\quad$ I is identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow x = 0$

$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow c_1 = 0, c_2 = 0$ is the only solution.

So, $\{(1,3), (3,5)\}$ is linearly independent.

Exercise 1. Show that the set $\{(2,3,1), (2,1,3), (1,1,1)\}$ is linearly dependent in $\mathbb{R}^3$.

2. Show that the set of vectors $\{(1,2,2), (2,1,2), (2,2,1)\}$ is linearly independent.

Subspace : Let V be a vector space over the field F. Let W be a non-empty subset of V. W is said to be a subspace of V if W forms a vector space over F with respect to the addition and scalar multiplication restricted to W and $F \times W$.

Examples 1 Let V be a vector space over a field F. Then V is itself a subspace of V. This subspace is called the improper subspace of V.

The set $\{\theta\}$ consisting only the null element $\theta \in V$ forms a subspace of V. It is called the trivial subspace of V.

2. Let S be a subset of $\mathbb{R}^2$ defined by $S = \{(x,y) \in \mathbb{R}^2 : y = 2x\}$

Then S is a non-empty subset of S as $(0,0) \in S$ and S is a subspace of R^2

We state a theorem without proof:

Theorem 2 A non-empty set W of a vector space V over a field F is a subspace of V if and only if

(i) $\alpha, \beta \in W \Rightarrow \alpha + \beta \in W$

(ii) $\alpha \in W,\ c \in F \Rightarrow c\alpha \in W$

Example 3 Show that $S = \{(x,y) \in \mathbb{R}^2 : 3x + 2y = 0\}$ is a subspace of $\mathbb{R}^2$

Proof : S is non-empty as $(0,0) \in S$

Let $\alpha, \beta \in S$ then $\alpha = (x_1, y_1)$ and $\beta = (x_2, y_2)$ such that $3x_1 + 2y_1 = 0$ and $3x_2 + 2y_2 = 0$ — (1)

Now $\alpha + \beta = (x_1 + x_2,\ y_1 + y_2)$

Now from (1), we have $3(x_1 + x_2) + y_1 + y_2$

$= (3x_1 + y_1) + 3x_2 + y_2 = 0 + 0 = 0$

So, $\alpha + \beta \in S$

Let $c \in \mathbb{R}$ and $\alpha \in S$, So $\alpha = (x, y)$ and $3x + y = 0$

Now $c\alpha = (cx, cy)$ Now $3cx + 3cy = c(3x + y) = c.0$
$= 0$

So, $c\alpha \in S$.

Hence by Theorem 2, S is a subspace of $\mathbb{R}^2$