Example 4 In $\mathbb{R}^3$, $\alpha = (4,3,5)$, $\beta = (0,1,3)$ and $\gamma = (2,1,1)$. Examine if α is a linear combination of β and γ

Solution: Let $\alpha = c\beta + d\gamma$ where $c, d \in \mathbb{R}$.

Then $(4,3,5) = c(0,1,3) + d(2,1,1) = (0+2d, c+d, 3c+d)$

So, $2d = 4$, $c+d = 3$, $3c+d = 5$, giving $c = 1, d = 2$

Hence $\alpha = \beta + 2\gamma$ and α is a linear combination of β and γ

Example 5 Examine if the set S is a subspace of $\mathbb{R}^3$ where

(i) $S = \{(x,y,z) \in \mathbb{R}^3 : x = 0\}$

(ii) $S = \{(x,y,z) \in \mathbb{R}^3 : x = 1\}$

(iii) $S = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 0\}$

(iv) $S = \{(x,y,z) \in \mathbb{R}^3 : xy = z\}$

Solution: (i) Let $\alpha, \beta \in S$ then $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$ and $x_1 = 0$ and $x_2 = 0$. Here S is non-empty as $(0,0,0) \in S$.

Let $c \in \mathbb{R}$ now $\alpha + \beta = (x_1+x_2, y_1+y_2, z_1+z_2)$

and $x_1 + x_2 = 0 + 0 = 0$ So, $\alpha + \beta \in S$

$c\alpha = (cx_1, cy_1, cz_1)$ and $cx_1 = c.0 = 0$

So, $c\alpha \in \mathbb{R}^3$

So, S is a subspace of $\mathbb{R}^3$

(ii) $(1,0,2) \in S$

but $2(1,0,2) = (2,0,4) \notin S$ as 2, the first component is $\neq 1$. So, S is not a subspace of $\mathbb{R}^3$

(iii) Let $\alpha, \beta \in S \Rightarrow \alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$ and
$x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$

Here S is non-empty as $(0,0,0) \in S$

Let $c \in \mathbb{R}$. Now $\alpha + \beta = (x_1+x_2, y_1+y_2, z_1+z_2)$

and $(x_1+x_2) + (y_1+y_2) + (z_1+z_2) = (x_1+y_1+z_1) + (x_2+y_2+z_2)$
$= 0 + 0 = 0$

So, $\alpha + \beta \in S$ and $c\alpha = (cx_1, cy_1, cz_1)$ and $cx_1 + cy_1 + cz_1 = c(x_1+y_1+z_1)$

$= c.0 = 0$ So, $c\alpha \in S$. So, S is a subspace of $\mathbb{R}^3$

(iv) $(1,1,2) \in S$ as $1.2 = 2$ and $(2,3,6) \in S$ as $2.3 = 6$

Now $(1,2,2) + (2,3,6) = (3,5,8)$ but ~~5x~~ $3.5 \neq 8$

So, $(3,5,8) \notin S$. So, S is not a subspace of $\mathbb{R}^3$.

Exercises 1. Prove that in a ~~real~~ vector space V over a field F

(i) $c(\alpha - \beta) = c\alpha - c\beta$ where $c \in F$, $\alpha, \beta \in V$ [$\alpha - \beta$ is defined as $\alpha + (-\beta)$]

(ii) $(c-d)\alpha = c\alpha - d\alpha$ where $c, d \in F$, $\alpha \in V$

(iii) $(-c)\alpha = -(c\alpha) = c(-\alpha)$ where $c \in F$ and $\alpha \in V$

2. Examine if the S is subspace of $\mathbb{R}^3$

(i) $S = \{(x,y,z) \in \mathbb{R}^3 : 2x + 3y + 4z = 0\}$

(ii) $S = \{(x,y,z) \in \mathbb{R}^3 : x + y = z\}$

(iii) $S = \{(x,y,z) \in \mathbb{R}^3 : x + y + z = 6\}$

(iv) $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$

3. If $\alpha = (1,1,2)$, $\beta = (0,2,1)$ and $\gamma = (2,2,4)$, examine ~~whether~~ if α is linear combination of β and γ.

Theorem 3 The intersection of two subspaces of a vector space V over a field F is a subspace of V

Proof: Let W_1 and W_2 be two subspaces of V.

$W_1 \cap W_2$ is non-empty as $\theta \in W_1 \cap W_2$ (θ is the null element in V)

Let $\alpha_1, \alpha_2 \in W_1 \cap W_2 \Rightarrow \alpha_1, \alpha_2 \in W_1$ and $\alpha_1, \alpha_2 \in W_2$

$\Rightarrow \alpha_1 + \alpha_2 \in W_1$ and as W_1 is a subspace

and $\alpha_1 + \alpha_2 \in W_2$ as W_2 is a subspace

$\Rightarrow \alpha_1 + \alpha_2 \in W_1 \cap W_2$

Let $c \in F$ and $\alpha \in W_1 \cap W_2$. So, $\alpha \in W_1$ and $\alpha \in W_2$

$\Rightarrow c\alpha \in W_1$ as W_1 is a subspace

and $c\alpha \in W_2$ as W_2 is a subspace

So, $c\alpha \in W_1 \cap W_2$. So, $W_1 \cap W_2$ is a subspace of V.

Note: The union of two subspaces may not be a subspace.

For example, let $S_1 = \{(x,y) \in \mathbb{R}^2 : x = 0\}$ and $S_2 = \{(x,y) \in \mathbb{R}^2 : y = 0\}$

Then S_1 and S_2 are two subspaces of $\mathbb{R}^2$

Now $(0,1) \in S_1$ and $(1,0) \in S_2$ But $(0,1) + (1,0) = (1,1) \notin S_1 \cup S_2$

So, $S_1 \cup S_2$ is not a subspace of $\mathbb{R}^2$

Some important subspaces of a vector space.

1. Let V be a vector space over a field F and let $\alpha \in V$.

Then the set $W = \{c\alpha : c \in F\}$ forms a subspace of V

Let $\alpha_1, \alpha_2 \in W$. Then

$\alpha_1 = c_1\alpha$ and $\alpha_2 = c_2\alpha$ for some $c_1, c_2 \in F$

Then $\alpha_1 + \alpha_2 = c_1\alpha + c_2\alpha = (c_1 + c_2)\alpha \in W$, since $c_1 + c_2 \in F$

So, $\alpha_1 + \alpha_2 \in W$

Let $c \in F$ and $\beta \in W$. Then $\beta = c_1\alpha$ for some $c_1 \in F$

Now $c\beta = c(c_1\alpha) = (cc_1)\alpha \in W$ as $cc_1 \in F$

So, $c\beta \in W$

So, W is a subspace of V

W is said to be the subspace generated by $\{\alpha\}$

2. Let V be a vector space over a field F and let $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$ Then the set $W = \{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n : c_i \in F, i = 1, 2, \ldots, n\}$ is a subspace of V

Let $\alpha, \beta \in W$ Then $\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$ for some $c_1, c_2, \ldots, c_n \in F$

and $\beta = d_1\alpha_1 + d_2\alpha_2 + \cdots + d_n\alpha_n$ for some $d_1, d_2, \ldots, d_n \in F$

Now $\alpha + \beta = (c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n) + (d_1\alpha_1 + d_2\alpha_2 + \cdots + d_n\alpha_n)$

$= (c_1 + d_1)\alpha_1 + (c_2 + d_2)\alpha_2 + \cdots + (c_n + d_n)\alpha_n \in W$

as $c_i + d_i \in F$, $i = 1, 2, \ldots, n$

So, $\alpha + \beta \in W$

Let $c \in F$ and $\alpha \in W$. So $\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$ for some $c_1, c_2, \ldots, c_n \in F$ Then $c\alpha = c(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n)$

$= (cc_1)\alpha_1 + (cc_2)\alpha_2 + \cdots + (cc_n)\alpha_n \in W$ as $cc_i \in F$, $i = 1, 2, \ldots, n$

So, $c\alpha \in W$. So, W is a subspace of V.

W is said to be the subspace generated by $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$

Note: W is the smallest subspace of V containing $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and is denoted by $L(S)$. It is also said to be the subspace generated by $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. W is sometimes also called linear span of S.

Example 1. Let $\alpha_1 = (1,0)$, $\alpha_2 = (0,1)$. Let $S = \{\alpha_1, \alpha_2\}$

Then $L(S)$ is a subspace of $\mathbb{R}^2$ and actually

$L(S) = \mathbb{R}^2$ as $(x,y) \in \mathbb{R}^2 \Rightarrow (x,y) = x(1,0) + y(0,1)$

So, $(x,y) \in L(S)$. So, $L(S) = \mathbb{R}^2$

Example 2 Let $\alpha = (1,0,0)$, $\beta = (0,1,0)$ $\gamma = (0,0,1)$

Let $S = \{\alpha, \beta, \gamma\}$ Then $L(S) = \mathbb{R}^3$

as $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

Theorem 4 Any superset of a linearly dependent set is linearly dependent.

Proof: Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a finite set of vectors in a vector space over a field F. Let S be linearly dependent. Let P be the set given by $P = \{\alpha_1, \alpha_2, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n\}$.

As S is linearly dependent, there exists scalars $c_1, c_2, \ldots, c_r$ not all zero in F such that

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r = \theta \quad \ldots \text{(i)}$$

From (i) we can write,

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r + 0.\alpha_{r+1} + \cdots + 0.\alpha_n = \theta \quad \text{— (ii)}$$

So, there exist scalars $c_1, c_2, \ldots, c_r, \overbrace{0, 0, \ldots, 0}^{(n-r)}$ not all zero such that (ii) holds.

So, P is linearly dependent

Corollary 4: Any subset of a linearly independent set is linearly independent.

Proof: Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a linearly independent set. Let $T \subset S$. If T is not linearly independent. So, T is linearly dependent and S is a superset of T. So, ~~is~~ S is linearly dependent, a contradiction. Hence T is linearly ~~indepent~~ independent.

Example 1. Examine if the set of vectors $\{(2,1,1), (1,2,2), (1,1,1)\}$ is linearly dependent in $\mathbb{R}^3$.

Solution: Let $\alpha = (2,1,1)$, $\beta = (1,2,2)$, $\gamma = (1,1,1)$

Let us consider the relation $c_1\alpha + c_2\beta + c_3\gamma = (0,0,0)$ where c_1, c_2, c_3 are real numbers. Then $c_1(2,1,1) + c_2(1,2,2) + c_3(1,1,1) = (0,0,0)$

So, $2c_1 + c_2 + c_3 = 0$, $c_1 + 2c_2 + c_3 = 0$, $c_1 + 2c_2 + c_3 = 0$

or, equivalently $2c_1 + c_2 + c_3 = 0$, $c_1 + 2c_2 + c_3 = 0$.

So, here $\left.\begin{array}{l} 2c_1 + c_2 + c_3 = 0 \\ \text{and } c_1 + 2c_2 + c_3 = 0 \end{array}\right\}$ This gives $\dfrac{c_1}{-1} = \dfrac{c_2}{-1} = \dfrac{c_3}{3} = k$ (say),

k is a real number. So, the solution is $c_1 = -k$, $c_2 = -k$, $c_3 = 3k$ where k is a real number. Since k is arbitrary, there exist c_1, c_2, c_3 not all zero, such that $c_1\alpha + c_2\beta + c_3\gamma = (0,0,0)$. For example, take $c_1 = 1$, $c_2 = 1$, $c_3 = -3$ Then $\alpha + \beta - 3\gamma = (0,0,0)$. So, the set is linearly dependent.

Example 2. Prove that the set of vectors $\{(1,2,2), (2,1,2), (2,2,1)\}$ is linearly independent

Solution: Let $\alpha = (1,2,2)$, $\beta = (2,1,2)$, $\gamma = (2,2,1)$

Let us consider the relation $c_1\alpha + c_2\beta + c_3 = (0,0,0)$ where c_1, c_2, c_3 are real numbers

Then $c_1(1,2,2) + c_2(2,1,2) + c_3(2,2,1) = (0,0,0)$

So, $c_1 + 2c_2 + 2c_3 = 0$, $2c_1 + c_2 + 2c_3 = 0$, $2c_1 + 2c_2 + c_3 = 0$

This is a homogeneous system of three equations in c_1, c_2, c_3.

The coefficient determinant of the system is $\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 5 \neq 0$

By, Cramer's rule, there exists a unique solution for c_1, c_2, c_3 and the solution is $c_1 = c_2 = c_3 = 0$. This proves that the set of vectors is linearly independent.