Theorem 5 If S and T be two non-empty finite subsets of a vector space V over a field F and $S \subset T$. Then $L(S) \subset L(T)$

Proof: Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and let $\alpha \in L(S)$

Then $\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$ for some $c_i \in F$, $i = 1, 2, \ldots, n$

As $\alpha_i \in S$, $i = 1, 2, \ldots, n$, so $\alpha_i \in T$, $i = 1, 2, \ldots, n$ as $S \subset T$

So, $c_i\alpha_i \in L(T)$ as $L(T)$ is a subspace of V, $i = 1, \ldots n$

So, $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n \in L(T)$ as $L(T)$ is a subspace of V

So, $\alpha \in L(T)$. So, $L(S) \subset L(T)$

We state another theorem without proof:

Theorem 6 If S and T be two non-empty finite subsets of a vector space V over a field F and each element of T is a linear combination of the vectors of S, then $L(T) \subset L(S)$.

Example 1 Determine the subspace of $\mathbb{R}^3$ spanned by the vectors $\alpha = (1, 2, 3)$, $\beta = (3, 1, 0)$. Examine if

(i) $\gamma = (2, 1, 3)$ is in the subspace

(ii) $\delta = (-1, 3, 6)$ is in the subspace.

Solution: The subspace spanned by the vectors α and β is

$$L(\{\alpha, \beta\}) = \{c\alpha + d\beta : c, d \in \mathbb{R}\}$$
$$= \{c(1,2,3) + d(3,1,0) : c, d \in \mathbb{R}\}$$
$$= \{(c+3d, 2c+d, 3c) : c, d \in \mathbb{R}\}$$

(i) If $\gamma \in L(\{\alpha, \beta\})$, then there must be $c, d \in \mathbb{R}$ such that

$$\gamma = c\alpha + d\beta \quad \text{or,} \quad (2, 1, 3) = (c+3d, 2c+d, 3c)$$

So, $c + 3d = 2$, $2c + d = 1$, $3c = 3$. These equations are inconsistent, and so γ is not in $L(\{\alpha, \beta\})$

(ii) If $\delta \in L(\{\alpha, \beta\})$ there must be $c, d \in \mathbb{R}$ such that

$\delta = c\alpha + d\beta \Rightarrow (-1, 3, 6) = (c+3d, 2c+d, 3c)$

$\Rightarrow c + 3d = -1$, $2c + d = 3$ and $3c = 6$, giving $c = 2$, $d = -1$

So, $\delta = 2(1,2,3) - (1,3,0)$, showing $\delta \in L(\{\alpha,\beta\})$

Example 2 In $\mathbb{R}^2$, $\alpha = (3,1)$, $\beta = (2,-1)$. Determine $L(\{\alpha,\beta\})$ and show that $L(\{\alpha,\beta\}) = \mathbb{R}^2$

Solution: Here $L(\{\alpha,\beta\}) = \{c\alpha + d\beta : c, d \in \mathbb{R}\}$

$= \{(3c+2d, c-d) : c, d \in \mathbb{R}\}$

As $L(\{\alpha,\beta\})$ is a subspace of $\mathbb{R}^2$. So, $L(\{\alpha,\beta\}) \subset \mathbb{R}^2$

Now let $(x,y) \in \mathbb{R}^2$

Let $(x,y) = c\alpha + d\beta$ for some $c, d \in \mathbb{R}$

Then $3c+2d = x \quad c-d = y$

giving $c = \frac{x+2y}{5}$, $d = \frac{x-2y}{5}$

So, $(x,y) \in L(\{\alpha,\beta\})$

Hence $\mathbb{R}^2 \subset L(\{\alpha,\beta\})$. So, $L(\{\alpha,\beta\}) = \mathbb{R}^2$

Example 3 Examine if the set S is a subspace of the vector space $\mathbb{R}_{2\times2}$, where

(i) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times2} : a+b = 0 \right\}$

(ii) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times2} : \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \right\}$

(iii) S is the set of all 2×2 real symmetric matrices

Solution: (i) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$. So, S is non-empty.

Let $A, B \in S \Rightarrow A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ $B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$, $a_1+b_1 = 0$, $a_2+b_2 = 0$

$A+B = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}$ and $(a_1+a_2) + (b_1+b_2) = (a_1+b_1) + (a_2+b_2) = 0+0 = 0$

So, $A+B \in S$. Let $k \in \mathbb{R}$ and $A \in S$

Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a+b = 0$. So, $kA = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$

and ~~$ka + k(b) = k$~~ $ka + kb = k(a+b) = k.0 = 0$

So $kA \in S$. So, S is a subspace of $\mathbb{R}_{2\times2}$

(ii) $\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \in S$ as $\begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} = 0$ and $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \in S$ as $\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$

Now $\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 2 & 2 \end{pmatrix}$ does not belong to S

as $\begin{vmatrix} 4 & 5 \\ 2 & 2 \end{vmatrix} = 8 - 10 = -2 \neq 0$. So, S is not a

subspace of $\mathbb{R}_{2\times 2}$

(iii) Let $A, B \in S$. Then A, B are symmetric. So, $A^t = A$, $B^t = B$

Now $(A+B)^t = A^t + B^t = A + B$. So, $A+B$ is symmetric.

Hence, $A+B \in S$

Let $c \in \mathbb{R}$ and $A \in S$. So, A is symmetric. So, $A^t = A$

Now $(cA)^t = cA^t = cA$. So, cA is symmetric

So, $cA \in S$. Hence S is a subspace of $\mathbb{R}_{2\times 2}$

Exercises: 1. In $\mathbb{R}^3$, $\alpha = (1,3,0)$, $\beta = (2,1,-2)$. Determine $L(\{\alpha,\beta\})$. Examine if $\gamma = (-1,3,2)$, $\delta = (4,7,-2)$ are in $L(\{\alpha,\beta\})$.

2. Examine if the set S is a subspace of the vector space of $\mathbb{R}_{2\times 2}$, where

(i) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times 2} : a + 2b = 0 \right\}$

(ii) S is the set of all 2x2 real skew-symmetric matrices

Basis and dimension of a vector space

Let V be a vector space over a field F. V is said to be finitely generated or finite dimensional if there exists a finite set of vectors S of V such that $L(S) = V$. Otherwise V is said to be infinite dimensional.

Example: consider the vector space $\mathbb{R}^2$ and the set $S = \{(1,0),(0,1)\}$. Then $L(S) = \mathbb{R}^2$. So, $\mathbb{R}^2$ is finite dimensional.

Basis: Let V be a finite dimensional vector space. A finite set S of V is said to be a basis of V if

(i) S is linearly independent (ii) $L(S) = V$

~~The number of elements in the basis is called~~

Example: Consider the finite dimensional vector space $\mathbb{R}^3$

Consider the set $S = \{(1,0,0),(0,1,0),(0,0,1)\}$

As $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$,

$L(S) = \mathbb{R}^3$

Let $c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$

$\Rightarrow (c_1, c_2, c_3) = (0,0,0)$

$\Rightarrow c_1 = c_2 = c_3 = 0$

So, $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly independent.

Hence S is a basis of $\mathbb{R}^3$

Some problems: 1. Show that the $S = \{(1,2),(3,1)\}$ is a basis of $\mathbb{R}^2$. ~~Let $(x,y) \in$~~

Solution: Let $(x,y) \in \mathbb{R}^2$ and assume that

$$(x,y) = c(1,2) + d(3,1)$$

$$\Rightarrow \quad c + 3d = x \quad 2c + d = y$$

giving $c = \frac{3y-x}{5}$, $d = \frac{2x-y}{5}$

Hence $L(S) = \mathbb{R}^2$

Let $c(1,2) + d(3,1) = 0$

So, $c + 3d = 0$

$2c + d = 0$

As, $\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0$

So, by Cramer's the only solution is $c = 0, d = 0$

So S is linearly independent

Hence S is a basis of $\mathbb{R}^2$

2. Prove that the set $S = \{(0,1,1), (1,0,1), (1,1,0)\}$ is a basis of $\mathbb{R}^3$.

Solution:

Let $c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0) = (0,0,0)$

Then $c_2 + c_3 = 0$, $c_1 + c_3 = 0$, $c_1 + c_2 = 0$

The coefficient matrix of the homogeneous system is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ $R_2' \to R_2 - R_3$

$= \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1+1 = 2 \neq 0$

Hence by Cramer's rule the system has unique solution $c_1 = c_2 = c_3 = 0$

Hence S is linearly independent

Now, let $(x, y, z) \in \mathbb{R}^3$

and assume $(x,y,z) = a(0,1,1) + b(1,0,1) + c(1,1,0)$

Then $b + c = x$, $a + c = y$, and $a + b = z$

giving $a = \dfrac{-x+y+z}{2}$, $b = \dfrac{x-y+z}{2}$, $c = \dfrac{x+y-z}{2}$

Hence $L(S) = \mathbb{R}^3$

Hence S is a basis of $\mathbb{R}^3$.

Exercises: 1. Prove that $\{(1,0), (2,3)\}$ is a basis of $\mathbb{R}^2$

2. Prove that $\{(3,5), (0,1)\}$ is a basis of $\mathbb{R}^2$

3. Is $\{(1,3), (2,6)\}$ is a basis of $\mathbb{R}^2$?

4. Prove that $\{(2,0,0), (0,3,0), (0,0,5)\}$ is a basis of $\mathbb{R}^3$

Note: We can prove that there may be different basis of a finite dimensional vector space. But the number of elements in any basis is same. This number is called the dimension of the vector space.