$\bar{C} = (\bar{c}_{ij})_{n\times n}$, $i=1,2,\ldots,n$, $j=1,2,\ldots,n$ be the new cost matrix

where $\bar{c}_{ij} = c_{ij} \pm u_i \pm v_j$ for all $i, j$, $u_i, v_j$ are constants

Thus we have added or subtracted the constants $u_i$ and $v_j$ from every element of the $i$th row and $j$th column of the cost matrix $C$ respectively.

Now for the new cost matrix $\bar{C}$, the new objective function $\bar{z}$ is given by

$$~~\bar{z} = \sum_{i=1}^{n}\sum_{j=1}^{n} \bar{c}_{ij}~~ \quad \bar{z} = \sum_{i=1}^{n}\sum_{j=1}^{n} \bar{c}_{ij} x_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} (c_{ij} \pm u_i \pm v_j) x_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} \pm \sum_{i=1}^{n}\sum_{j=1}^{n} u_i x_{ij} \pm \sum_{i=1}^{n}\sum_{j=1}^{n} v_j x_{ij}$$

$$= z \pm \sum_{i=1}^{n} u_i \sum_{j=1}^{n} x_{ij} \pm \sum_{j=1}^{n} v_j \sum_{i=1}^{n} x_{ij}$$

$$= z \pm \lambda \pm \mu \quad \left[ \text{Since } \sum_{j=1}^{n} x_{ij} = 1 = \sum_{i=1}^{n} x_{ij} \right]$$

and where $\lambda = \sum_{i=1}^{n} u_i$ = constant

and $\mu = \sum_{j=1}^{n} v_j$ = constant

Since $\lambda$ and $\mu$ are constants, so that assignment $x_{ij} = \bar{x}_{ij}$ which will minimize $z$ will also minimize $\bar{z}$.

Thus the theorem is proved.

Theorem 2 If in any assignment problem, all $c_{ij} \geq 0$, then a set of solution $x_{ij} = \bar{x}_{ij}$ which satisfies $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\bar{x}_{ij} = 0$ will be an optimal solution.

Proof: Since for $x_{ij} = \bar{x}_{ij}$, $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\bar{x}_{ij} = 0$, So, $\bar{z} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} = 0$ and this is the minimum value of $z$. So, the solution $x_{ij} = \bar{x}_{ij}$ satisfying the condition $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} = 0$ will be an optimal solution to the problem. Hence the theorem.

The above two theorem suggests that by adding or subtracting selected constants $u_i, v_j$ from rows and columns we can form a cost matrix $\bar{C}$ with elements $\bar{c}_{ij}$ such that $\bar{c}_{ij} \geq 0$ for all $i$ and $j$ and it will not affect the optimal solution. Now we choose $u_i$ and $v_j$ in such a way there will be at least one $\bar{c}_{ij} = 0$ in each row and each column. Then we will try to find a feasible assignment for which the corresponding $c_{ij}$ will be equal to zero. The assignment schedule will be optimal if there is exactly one assignment (i.e. exactly one assigned 0) in each row and each column, because, in that case, it will produce $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} = 0$, for assigned value of $x_{ij} = 1$ and the other values of $x_{ij}$ are zero.

Algorithm for the Hungarian Method

Algorithm that we follow to find an optimal assignment is ~~Hungarian~~ algorithm for the Hungarian method which is as follows:

Step 1 In each row of the cost matrix, locate the smallest element (row's minimum) and subtract it from every element in that row. From the resulting matrix, identify the smallest element of each column (column's minimum)

and subtract it from every element in that column.
The revised cost matrix will have at least one zero in every row and every column. If there is one zero in any row or any column, that zero is generally put with □ for identification and is called assigned zero.

Step 2 If the revised matrix has n zero entries only and no two of which are in the same row or column, then the assignment at these assigned zero cells is optimal. Otherwise follow step 3

Step 3 If at least one row or column of the revised cost matrix contains more than one zero entries, then draw the minimum number of horizontal and vertical lines to cover all zeros in the revised cost matrix. Each horizontal line must cover entire row and similar for each vertical line.

Now two cases may arise:
(a) The minimum number of lines so drawn is equal to the order of the cost matrix and in this case, optimal assignment is possible an we will follow step 4.
(b) The minimum number of lines so drawn be less than the order of the cost matrix and then we follow step 5.

Step 4 Examine the rows of the revised matrix starting from the first row. If any row contains only one zero, there will be made an assignment and so put this zero within □. When all the rows are checked, draw vertical lines along those columns containing assigned zeros.

Then follow the same procedure for the uncrossed columns one after another.

If there be only one zero in any column, an assignment is to ~~made~~ be made at that zero and put the zero with □ as before. When all uncrossed columns are checked draw horizontal lines along those rows which contains assigned zeros. Finally, we get all assigned zeros by allocating arbitrarily in the position of zeros not covered by the lines.

If there remains more than one zero in a row or in a column uncovered, we make assignments for these zeros by trial and error method and in this case alternative solution may exist. It is to be noted there will be only one assigned zero in every row and every column for optimal solution and the assignment will be made at these assigned zero cells only.

Step 5 ~~Ste~~ Select the smallest uncovered element in the above cost matrix. Subtract this element from every uncovered element and add it to every element at the intersection of two lines. Then go to step 3 with this modified cost matrix.

Repeat the steps 5 and 3 if needed and then apply 4.