Notes on GE 3 (My portion of the syllabus)

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Books followed : 1. Integral Calculus - Maity & Ghosh
2. Integral Calculus - Das & Mukherjee

Unit-1 Integral Calculus : My portion of the syllabus :

- Evaluation of definite integrals.
- Integration as the limit of a sum (with equally spaced as well as unequal intervals).
- Reduction formula of $\int \sin^n x \cos^m x\,dx$, $\int \frac{\sin^m x}{\cos^n x}\,dx$, $\int \tan^n x\,dx$ and associated problems (m and n are non-negative integers).
- Definition of Improper integrals : Statement of μ-test (ii) Comparison test (Limit form excluded). Simple problems only. Use of Beta and Gamma functions (convergence and ~~improper~~ important relations being assumed).

Working Knowledge of double integral.

AND

Unit-3 - ~~the whole portion of the syl~~ Linear Programming : Complete portion of the syllabus.

We start with the Unit-1 portion first.

Evaluation of definite integrals : For evaluation of definite integrals we recapitulate formulas for evaluation of definite integrals as it was in your syllabus in (10+2) system.

1. $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$ when $\phi'(x) = f(x)$

2. $\int_a^b f(x)\,dx = \int_a^b f(u)\,du$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ when $a < c < b$

4. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

5. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

6. $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$, when $f(a+x) = f(x)$, n is a positive integer.

7. $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{when } f(2a-x) = f(x) \\ 0, & \text{when } f(2a-x) = -f(x) \end{cases}$$

8. $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{when } f \text{ is an even function, i.e., } f(-x) = f(x) \\ 0, & \text{when } f \text{ is an odd function, i.e., } f(-x) = -f(x) \end{cases}$$

Some worked out exercises

1. Find the value of $\displaystyle\int_0^{\pi/4} \frac{\sin^2 x\cos^2 x}{\sin^3 x + \cos^3 x}\,dx$

Solution Let $I = \displaystyle\int_0^{\pi/4} \frac{\sin^2 x\cos^2 x}{\sin^3 x + \cos^3 x}\,dx$. Then

$$I = \int_0^{\pi/4} \frac{\sec^2 x}{(\tan^2 x + \cot x)^2}\,dx$$ dividing numerator and denominator by $\sin^2 x\cos^4 x$

$$= \int_0^{\pi/4} \frac{\tan^2 x\sec^2 x}{(\tan^3 x + 1)^2}\,dx$$ Let $\tan x = u$, so, $\sec^2 x\,dx = du$

and

x	0	$\pi/4$
u	0	1

So, $I = \displaystyle\int_0^1 \frac{u^2}{(u^3+1)^2}\,du$ Let $1+u^3 = t$, so, $3u^2\,du = dt$

and

u	0	1
t	1	2

So, $I = \dfrac{1}{3}\displaystyle\int_1^2 \frac{dt}{t^2} = \frac{1}{3}\left[-\frac{1}{t}\right]_1^2 = \frac{1}{3}\left[1 - \frac{1}{2}\right] = \frac{1}{6}$

2. Find $\displaystyle\int_0^{\pi/4} \cos^5 x\,dx$. ~~Let $I = \int_0^{\pi/4} (1-\sin^2 x)^2 \cos x\,dx$~~

Solution: Let $I = \displaystyle\int_0^{\pi/4} \cos^5 x$. Then $I = \displaystyle\int_0^{\pi/4} (1-\sin^2 x)^2 \cos x\,dx$

Let $\sin x = u$, so, ~~$\cos x\,d$~~ $\cos x\,dx = du$,

x	0	$\pi/4$
u	0	$\frac{1}{\sqrt{2}}$

So, $I = \displaystyle\int_0^{1/\sqrt{2}} (1-u^2)^2\,du = \int_0^{1/\sqrt{2}} (1 - 2u^2 + u^4)\,du = \left[u - \frac{2}{3}u^3 + \frac{u^5}{5}\right]_0^{1/\sqrt{2}}$

$= \dfrac{43}{60\sqrt{2}}$

3. Find $\int_0^{\pi/4} \cos^6 x \, dx$.

Solution $\int_0^{\pi/4} \cos^6 x \, dx = \frac{1}{8}\int_0^{\pi/4} (2\cos^2 x)^3 dx = \frac{1}{8}\int_0^{\pi/4} (1+\cos 2x)^3 dx$

$= \frac{1}{8}\int_0^{\pi/4} (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x) dx$

$= \frac{1}{8}\left[x + \frac{3}{2}\sin 2x\right]_0^{\pi/4} + \frac{3}{16}\int_0^{\pi/4}(1+\cos 4x) dx + \frac{1}{8}\int_0^{\pi/4}(1-\sin^2 2x)\cos 2x \, dx$

$= \frac{1}{8}\left(\pi/4 + \frac{3}{2}\right) + \frac{3}{16}\left[x + \frac{\sin 4x}{4}\right]_0^{\pi/4} + \frac{1}{16}\int_0^1 (1-u^2) du$ taking $u = \sin 2x$

$= \frac{1}{8}\left(\pi/4 + \frac{3}{2}\right) + \frac{3}{16}\left[\pi/4\right] + \frac{1}{16}\left[u - \frac{u^3}{3}\right]_0^1$

$= \frac{1}{32}(\pi + 6) + \frac{3\pi}{64} + \frac{1}{24} = \frac{15\pi + 44}{192}$

4. Find $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx = \int_0^{\pi/2} \sin^4 x (1-\sin^2 x)^2 \cos x \, dx$

$= \int_0^1 u^4 (1-u^2)^2 du$, taking $u = \sin x$

$= \left[\frac{u^5}{5} - \frac{2}{7}u^7 + \frac{u^9}{9}\right]_0^1 = \frac{8}{315}$

5. Prove that $\int_a^b g(x) dx = \int_a^b g(a+b-x) dx$

Proof: Let $a+b-x = u$. So, $-dx = du$,

x	a	b
u	b	a

So, $\int_a^b g(a+b-x) dx = -\int_b^a g(u) du = \int_a^b g(u) du = \int_a^b g(x) dx$

as $-\int_a^b f(x) dx = \int_b^a f(x) dx$ and $\int_a^b f(x) dx = \int_a^b f(u) du$.

6. Show that $\int_0^2 |1-x| dx = 1$ and $\int_{-1}^2 |x| dx = \frac{5}{2}$

Solution: $\int_0^2 |1-x| dx = \int_0^1 (1-x) dx + \int_1^2 (x-1) dx$ $\left[\text{As } |x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}\right]$

$= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2 = \frac{1}{2} + \frac{3}{2} - 1 = 1$

$$\int_{-1}^{2}|x|\,dx = \int_{-1}^{0}(-x)\,dx + \int_{0}^{2}x\,dx = -\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2} = \frac{5}{2}$$

7. Show that $\int_0^{\pi/2}(a^2\cos^2x + b^2\sin^2x)\,dx = \frac{\pi}{4}(a^2+b^2)$

Solution: Let $I = \int_0^{\pi/2}(a^2\cos^2x + b^2\sin^2x)\,dx = \int_0^{\pi/2}\left(a^2\cos^2(\pi/2-x) + b^2\sin^2(\pi/2-x)\right)dx$

$\left[\text{As } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$= \int_0^{\pi/2}(a^2\sin^2x + b^2\cos^2x)\,dx$$

Then $2I = \int_0^{\pi/2}\left(a^2(\cos^2x+\sin^2x) + b^2(\sin^2x+\cos^2x)\right)dx$

$$= (a^2+b^2)\int_0^{\pi/2}dx = \frac{\pi}{2}(a^2+b^2)$$

So, $I = \frac{\pi}{4}(a^2+b^2)$

8. Show that $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$ [Here we write $\log x = \ln x = \log_e x$]

Solution: Let $x = \tan\theta$. So, $dx = \sec^2\theta\,d\theta$,

x	0	1
θ	0	$\pi/4$

Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$. So, $I = \int_0^{\pi/4}\frac{\log(1+\tan\theta)}{\sec^2\theta}\cdot\sec^2\theta\,d\theta$

$$= \int_0^{\pi/4}\log(1+\tan\theta)\,d\theta = \int_0^{\pi/4}\log\left(1+\tan(\pi/4-\theta)\right)$$

$$= \int_0^{\pi/4}\log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta = \int_0^{\pi/4}\log\frac{2}{1+\tan\theta}\,d\theta$$

$$= \log 2\int_0^{\pi/4}d\theta - \int_0^{\pi/4}\log(1+\tan\theta)\,d\theta$$

So, $2I = \frac{\pi}{4}\log 2$ or, $I = \frac{\pi}{8}\log 2$

Exercise: Show that

(i) $\int_0^{\pi}\frac{x\,dx}{a^2\cos^2x + b^2\sin^2x} = \frac{\pi^2}{2ab}$ (ii) $\int_0^{\pi}\frac{x\,dx}{(a^2\cos^2x+b^2\sin^2x)^2} = \frac{\pi(a^2+b^2)}{4a^3b^3}$

(iii) $\int_0^{\pi/2}\frac{x\,dx}{\sec x + \csc x} = \frac{\pi}{4}\left(1+\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)\right)$