Integration as the limit of a sum (with equally spaced as well as unequal intervals). We recapitulate the following definitions:

Definition of definite integral: $\int_a^b f(x)\,dx = \lim_{h\to 0+} h \sum_{r=0}^{n-1} f(a+rh)$

$= \lim_{h\to 0+} h \sum_{r=1}^{n} f(a+rh) = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=0}^{n-1} f\left(a+\frac{b-a}{n}r\right) = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a+\frac{b-a}{n}r\right)$

Here we partitioned the interval $[a,b]$ in n equal subintervals of lenth h. So, $b-a = nh$ ~~as~~. The partition is shown in Fig 1

Another definition of definite integral $\int_a^b f(x)\,dx$. We take a partition $\{a, a\mu, a\mu^2, \dots, a\mu^n\}$ of $[a,b]$ so that $b = a\mu^n$ or, $\mu = \left(\frac{b}{a}\right)^{\frac{1}{n}}$. Length of the rth subinterval $[a\mu^{r-1}, a\mu^r]$ is $\delta_r = a\mu^r - a\mu^{r-1} = a\mu^{r-1}(\mu-1)$, choose $\xi_r \in [a\mu^{r-1}, a\mu^r]$ as $\xi_r = a\mu^{r-1}$. Then $\int_a^b f(x)\,dx = \lim_{\delta\to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$ where δ = maximum of $\delta_1, \delta_2, \dots \delta_n$ [$\delta \to 0$ has been taken on the assumption that $n\to\infty$ and then $\mu = \left(\frac{b}{a}\right)^{\frac{1}{n}} \to 1$]

(This is used for the unequally spaced intervals]

Worked out exercises: 1. Evaluate the integral $\int_a^b e^x\,dx$ by the method of summation

Solution. $\int_a^b e^x\,dx = \lim_{h\to 0+} h \sum_{r=1}^{n} e^{a+rh}$ $[nh = b-a]$

$= \lim_{h\to 0+} h\left(e^{a+h} + e^{a+2h} \dots + e^{a+nh}\right) = e^a \lim_{h\to 0+} h\left(e^h + e^{2h} + \dots + e^{nh}\right)$

$= e^a \lim_{h\to 0+} h e^h \frac{e^{nh}-1}{e^h-1} = e^a\left(e^{b-a}-1\right) \lim_{h\to 0+} \frac{he^h}{e^h-1}$ $[nh = b-a]$

$= \left(e^b - e^a\right) \lim_{h\to 0+} \frac{e^h}{\frac{e^h-1}{h}} = e^b - e^a$ $\left[\text{as } \lim_{h\to 0+} \frac{e^h-1}{h} = 1\right]$

2. Evaluate $\int_0^{\pi/2} \cos x \, dx$ by the method of summation.

Solution $\int_0^{\pi/2} \cos x \, dx = \lim_{h \to 0+} h \sum_{r=1}^{n} \cos rh \quad \left[nh = \pi/2 \right]$

$$= \lim_{h \to 0+} h \left(\cos h + \cos 2h + \cdots + \cos nh \right)$$

$$= \lim_{h \to 0+} h \frac{\sin nh/2}{\sin h/2} \cos\left(h + \frac{n-1}{2} h \right)$$

$\left[\text{Note: } \cos\alpha + \cos(\alpha+\beta) + \cdots + \cos(\alpha + (n-1)\beta) = \frac{\sin \frac{n\beta}{2}}{\sin \beta/2} \cos\left(\alpha + \frac{n-1}{2}\beta\right) \right]$

$$= \lim_{h \to 0} h \frac{\sin \frac{\pi}{4}}{\sin \frac{h}{2}} \cos\left(\frac{h}{2} + \pi/4 \right) \quad \left[\text{As } nh = \frac{\pi}{4} \right]$$

$$= \frac{1}{\sqrt{2}} \cdot 2 \lim_{h \to 0+} \frac{\frac{h}{2}}{\sin \frac{h}{2}} \cdot \lim_{h \to 0+} \cos\left(\frac{h}{2} + \frac{\pi}{4} \right)$$

$$= \sqrt{2} \cdot 1 \cdot \frac{1}{\sqrt{2}} = 1 \qquad \left[\text{As } \lim_{h \to 0+} \frac{\sin h}{h} = 1 \right]$$

3. Evaluate $\int_a^b \frac{1}{x} dx$ by the method of summation.

Solution: $f(x) = \frac{1}{x}$, $\xi_r = a\mu^{r-1}$, $\delta_r = a\mu^r - a\mu^{r-1} = a\mu^{r-1}(\mu - 1)$

and $b = a\mu^n$. When $n \to \infty$, $\delta \to 0$, $\mu \to 1$ where δ = maximum of $\delta_1, \delta_2, \ldots \delta_n$

Then $\int_a^b \frac{1}{x} dx = \lim_{\delta \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r = \lim_{\delta \to 0} \sum_{r=1}^{n} \frac{1}{\xi_r} \delta_r$

$$= \lim_{\mu \to 1} \sum_{r=1}^{n} \frac{1}{a\mu^{r-1}} a\mu^{r-1}(\mu - 1) = \lim_{\mu \to 1} n(\mu - 1) = \lim_{\mu \to 1} (\mu - 1) \frac{\log \frac{b}{a}}{\log \mu} \quad (\text{As } b = a\mu^n)$$

$$= \log \frac{b}{a} \lim_{\mu \to 1} \frac{\mu - 1}{\log \mu} \left[\frac{0}{0} \right] = \log \frac{b}{a} \lim_{\mu \to 1} \frac{1}{\frac{1}{\mu}} \quad (\text{By L'Hospital's rule})$$

$$= \log \frac{b}{a}$$

4. Evaluate $\int_a^b x^4 dx$ by the method of summation.

Solution: Here $f(x) = x^4$, $\xi_r = a\mu^{r-1}$, $\delta_r = a\mu^r - a\mu^{r-1} = a\mu^{r-1}(\mu-1)$ and $b = a\mu^n$. When $n \to \infty$, $\delta \to 0$, $\mu \to 1$ where δ = maximum of $\delta_1, \delta_2, \cdots \delta_n$.

Then $\int_a^b x^4 dx = \lim_{\mu \to 1} \sum_{r=1}^{n} (a\mu^{r-1})^4 a\mu^{r-1}(\mu-1) = a^5 \lim_{\mu\to 1} (\mu-1) \sum_{r=1}^{n} \mu^{5r-5}$

$$= a^5 \lim_{\mu\to 1} \frac{\mu-1}{\mu^5}\left(\mu^5 + \mu^{10} + \cdots + \mu^{5n}\right) = a^5 \lim_{\mu\to1} \frac{\mu-1}{\mu^5}\, \mu^5 \frac{(\mu^{5n}-1)}{\mu^5-1}$$

$$= a^5 \lim_{\mu\to1} (\mu-1) \frac{\left(\left(\frac{b}{a}\right)^5 - 1\right)}{\mu^5-1} = a^5\left(\left(\frac{b}{a}\right)^5 - 1\right) \cdot \lim_{\mu\to} \frac{1}{\frac{\mu^5-1}{\mu-1}}$$

$$= (b^5 - a^5) \cdot \frac{1}{5} = \frac{b^5 - a^5}{5}$$

5. Find the following limit:

$$\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3} + \frac{2^2}{n^3+2^3} + \cdots + \frac{n^2}{2n^3}\right]$$

Solution: $\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3} + \frac{2^2}{n^3+2^3} + \cdots + \frac{n^2}{2n^3}\right]$

$$= \lim_{n\to\infty} \frac{n^2}{n^3}\left[\frac{\left(\frac{1}{n}\right)^2}{1+\left(\frac{1}{n}\right)^3} + \frac{\left(\frac{2}{n}\right)^2}{1+\left(\frac{2}{n}\right)^3} + \cdots + \frac{\left(\frac{n}{n}\right)^2}{1+\left(\frac{n}{n}\right)^3}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3} = \int_0^1 \frac{x^2}{1+x^3}$$

$$= \frac{1}{3}\int_0^1 \frac{3x^2}{1+x^3} dx = \frac{1}{3}\left[\log|1+x^3|\right]_0^1 = \frac{1}{3}\log 2$$

6. Show that $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}\right] = \log 3$

Proof: $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n}\right]$

$= \lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}+\frac{1}{2n+1}+\frac{1}{2n+2}+\cdots+\frac{1}{2n+n}\right]$

$= 0+\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right]+\lim_{n\to\infty}\left[\frac{1}{2n+1}+\frac{1}{2n+2}+\cdots+\frac{1}{2n+n}\right]$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}}+\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{2+\frac{r}{n}}$

$= \int_0^1\frac{dx}{1+x}+\int_0^1\frac{dx}{2+x} = \left[\log_e|1+x|\right]_0^1+\left[\log|2+x|\right]_0^1$

$= \log 2+\log 3-\log 2 = \log 3$

7. Find $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n^2}{(n^2+r^2)^{3/2}}$

Solution: $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n^2}{(n^2+r^2)^{3/2}} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\left(1+\left(\frac{r}{n}\right)^2\right)^{3/2}}$

$= \int_0^1\frac{dx}{(1+x^2)^{3/2}}$. Let $x=\tan\theta$. So, $dx=\sec^2\theta\,d\theta$

and

x	0	1
θ	0	$\pi/4$

$= \int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \int_0^{\pi/4}\cos\theta\,d\theta = \left[\sin\theta\right]_0^{\pi/4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Exercise: 1. Find the following integrals by the method of summation:

(a) $\int_0^1\cos^3 x\,dx$ (b) $\int_0^{\pi/2}\sin x\,dx$

(c) $\int_a^b\sqrt{x}\,dx$ (d) $\int_a^b\frac{1}{x^2}\,dx$

2. Find the following limits:

(a) $\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\cdots+\frac{1}{n}\right]$

(b) $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{\sqrt{n^2-1^2}}{n^2}+\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$