Worked out exercises :

1. Show that $\lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2\left(1+\frac{3^2}{n^2}\right)^3\cdots\left(1+\frac{n^2}{n^2}\right)^n\right\}^{\frac{2}{n^2}} = \frac{4}{e}$

Solution: Let $u = \lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2\left(1+\frac{3^2}{n^2}\right)^3\cdots\left(1+\frac{n^2}{n^2}\right)^n\right\}^{\frac{2}{n^2}}$

Then $\log u = \frac{2}{n^2}\left[\log\left(1+\frac{1}{n^2}\right)+2\log\left(1+\frac{2^2}{n^2}\right)+3\log\left(1+\frac{3^2}{n^2}\right)+\cdots+n\log\left(1+\frac{n^2}{n^2}\right)\right]$

$= \frac{2}{n}\left[\frac{1}{n}\log\left(1+\frac{1^2}{n^2}\right)+\frac{2}{n}\log\left(1+\frac{2^2}{n^2}\right)+\frac{3}{n}\log\left(1+\frac{3^2}{n^2}\right)+\cdots+\frac{n}{n}\log\left(1+\frac{n^2}{n^2}\right)\right]$

$= \frac{2}{n}\sum_{r=1}^{n}\frac{r}{n}\log\left(1+\left(\frac{r}{n}\right)^2\right)$

So, $\lim_{n\to\infty}\log u = \lim_{n\to\infty}\frac{2}{n}\sum_{r=1}^{n}\frac{r}{n}\log\left(1+\left(\frac{r}{n}\right)^2\right)$

$= 2\int_0^1 x\log(1+x^2)\,dx$ Let $1+x^2 = v$, $2x\,dx = dv$

x	0	1
v	1	2

$= \int_1^2 \log v\,dv = [v\log v]_1^2 - \int_1^2 v\cdot\frac{1}{v}\,dv$

$= 2\log 2 - [v]_1^2 = 2\log 2 - 1 = \log\frac{4}{e}$

$\therefore \lim_{n\to\infty}\log u = \log\lim_{n\to\infty}u = \log\frac{4}{e}$ [As log function is continuous]

So, $\lim_{n\to\infty} u = \frac{4}{e}$

2. Show that $\lim_{n\to\infty}\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right\}^{\frac{1}{n}} = 2e^{\frac{\pi}{2}-2}$

Solution: Let $u = \left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$

So, $\log u = \frac{1}{n}\left[\log\left(1+\frac{1^2}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\cdots+\log\left(1+\frac{n^2}{n^2}\right)\right]$

So, $\log u = \frac{1}{n} \sum_{r=1}^{n} \log\left(1+\frac{r^2}{n^2}\right)$

So, $\lim_{n\to\infty} \log u = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \log\left(1+\left(\frac{r}{n}\right)^2\right)$

$$= \int_0^1 \log(1+x^2)\,dx$$

$$= \left[x\log(1+x^2)\right]_0^1 - \int_0^1 x\cdot\frac{1}{1+x^2}\cdot 2x\,dx$$

$$= \log 2 - 2\int_0^1 \frac{x^2}{1+x^2}\,dx = \log 2 - 2\int_0^1 \left(1-\frac{1}{1+x^2}\right)dx$$

$$= \log 2 - 2\left[x - \tan^{-1}x\right]_0^1 = \log 2 - 2\left(1-\frac{\pi}{4}\right)$$

$$= \log 2 + \frac{\pi}{2} - 2 = \log 2e^{\pi/2 - 2}$$

So, $\lim_{n\to\infty} \log u = \log \lim_{n\to\infty} u = \log 2e^{\pi/2-2}$ (As log function is continuous)

So, $\lim_{n\to\infty} u = 2e^{\pi/2-2}$

Exercise: 1. Show that $\lim_{n\to\infty} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right\}^{\frac{1}{n}} = \frac{4}{e}$

<u>Reduction formula</u>: Reduction formula is a type of recurrence relation or formula that may be express the integral of a complicated functions in terms of simpler ones. In this section, we find out reduction formulae for the functions $\sin^n x$, $\cos^n x$, $\tan^n x$ etc.

<u>Worked out examples</u>: 1. Find a reduction formula for $\int \sin^n x\,dx$, n being a positive integer and hence find $\int \sin^5 x\,dx$

Solution Let $I_n = \int \sin^n x \, dx$

$= \int \sin^{n-1} x \cdot \sin x \, dx = \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2}\cos x(-\cos x)\,dx$ [Integrating by parts]

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x \, dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x (1-\sin^2 x)\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x - (n-1)\int \sin^n x \, dx$

$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$

So, $I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

So, $n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

So, $I_n = -\dfrac{\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} I_{n-2}$ .... (1)

So, (1) is a reduction formula for $I_n = \int \sin^n x \, dx$

So, $\int \sin^5 x \, dx = I_5$. So, $I_5 = -\dfrac{\sin^4 x \cos x}{5} + \dfrac{4}{5} I_3$ from (1)

Again $I_3 = -\dfrac{\sin^2 x \cos x}{3} + \dfrac{2}{3} I_1$

Now $I_1 = \int \sin x \, dx = -\cos x$

So, $I_3 = -\dfrac{\sin^2 x \cos x}{3} - \dfrac{2}{3}\cos x$

So, $I_5 = -\dfrac{\sin^4 x \cos x}{5} + \dfrac{4}{5}\left(-\dfrac{\sin^2 x \cos x}{3} - \dfrac{2}{3}\cos x\right) + c$ [where c is the integration constant]

So, $\int \sin^5 x \, dx = -\dfrac{\sin^4 x \cos x}{5} - \dfrac{4}{15}\sin^2 x \cos x - \dfrac{8}{15}\cos x + c$

Exercise : 1. Find $\int \sin^7 x \, dx$ 2. $\int \sin^6 x \, dx$

Worked out Example : 2. Find a reduction formula for $\int \cos^n x\,dx$, n being a positive integer. Hence evaluate $\int_0^{\pi/2} \cos^n x\,dx$.

Solution: Let $I_n = \int \cos^n x\,dx = \int \cos^{n-1}x \cos x\,dx$

$= \cos^{n-1}x \sin x - \int (n-1)\cos^{n-2}x\,(-\sin x)\sin x\,dx$

$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x \sin^2 x\,dx$

$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,(1-\cos^2 x)\,dx$

$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$

So, $I_n = \cos^{n-1}x \sin x + (n-1)I_{n-2} - (n-1)I_n$

So, $nI_n = \cos^{n-1}x \sin x + (n-1)I_{n-2}$

or, $I_n = \dfrac{\cos^{n-1}x \sin x}{n} + \dfrac{n-1}{n} I_{n-2}$. This is a required reduction formula.

Now, let $J_n = \int_0^{\pi/2} \cos^n x\,dx$

So, $J_n = \left[\dfrac{\cos^{n-1}x \sin x}{n}\right]_0^{\pi/2} + \dfrac{n-1}{n} J_{n-2}$

$= 0 + \dfrac{n-1}{n} J_{n-2}$

$= \dfrac{n-1}{n} J_{n-2}$

So, $J_n = \begin{cases} \dfrac{n-1}{n}\,\dfrac{n-3}{n-2}\cdots\dfrac{2}{3}J_1 & \text{when } n \text{ is odd} \\ \dfrac{n-1}{n}\,\dfrac{n-3}{n-2}\cdots\dfrac{3}{4}J_2 & \text{when } n \text{ is odd} \end{cases}$

Now $J_1 = \int_0^{\pi/2}\cos x\,dx = [\sin x]_0^{\pi/2} = 1$ and $J_2 = \int_0^{\pi/2}\cos^2 x\,dx$

So, $J_2 = \frac{1}{2}\int_0^{\pi/2}(1+\cos 2x)\,dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \pi/4$ [or, $J_2 = \int_0^{\pi/2}\cos^2 x = \int_0^{\pi/2}\sin^2 x = \frac{1}{2}\int_0^{\pi/2}dx = \pi/4$]