So, $J_n = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3} & \text{when } n \text{ is odd} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\times\frac{\pi}{4} & \text{when } n \text{ is even} \end{cases}$ ... (1)

Note: 1. $\int_0^{\pi/2}\sin^n x = \int_0^{\pi/2}\sin^n(\pi/2 - x)\,dx = \int_0^{\pi/2}\cos^n x\,dx$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

2. $\int_0^{\pi/2}\cos^7 x\,dx = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{48}{105}$, $\int_0^{\pi/2}\cos^8 x\,dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{\pi}{4} = \frac{105\pi}{768}$ from (1)

3. $\int_0^{\pi/2}\sin^7 x\,dx = \int_0^{\pi/2}\cos^7 x\,dx = \frac{48}{105}$, $\int_0^{\pi/2}\sin^8 x\,dx = \int_0^{\pi/2}\cos^8 x\,dx = \frac{105\pi}{768}$

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3. Find a reduction formula for $\int \sin^m x\cos^n x$ ($m, n$ both being positive integers).

Solution: Let $I_{m,n} = \int \sin^m x\cos^n x\,dx$

Then $I_{m,n} = \int \cos^{n-1}x\,(\sin^m x\cos x)\,dx$

$$= \frac{\cos^{n-1}x\sin^{m+1}x}{(m+1)} - \frac{n-1}{m+1}\int \cos^{n-2}x\,(-\sin x)\sin^{m+1}x\,dx$$

$$= \frac{\cos^{n-1}x\sin^{m+1}x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2}x\sin^m x\,(1-\cos^2 x)\,dx$$

$$= \frac{\cos^{n-1}x\sin^{m+1}x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2}\sin^m x - \frac{n-1}{m+1}\int \cos^n x\sin^m x\,dx$$

$$= \frac{\cos^{n-1}x\sin^{m+1}x}{m+1} + \frac{n-1}{m+1}I_{m,n-2} - \frac{n-1}{m+1}I_{m,n}$$

Hence $\left(1+\frac{n-1}{m+1}\right)I_{m,n} = \frac{\cos^{n-1}x\sin^{m+1}x}{m+1} + \frac{n-1}{m+1}I_{m,n-2}$

So, $I_{m,n} = \frac{\cos^{n-1}x\sin^{m+1}x}{m+n} + \frac{n-1}{m+n}I_{m,n-2}$ ... (1)

This is one reduction formula for $I_{m,n}$. Another reduction formula of $I_{m,n}$ can be written similarly by taking

$I_{m,n} = \int \sin^{m-1}x\,(\cos^n x\sin x)\,dx$ and using integration by parts. Then the formula that we get is

$$I_{m,n} = \frac{\cos^{n+1}x \sin^{m-1}x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \quad \cdots (2)$$

4. Find a reduction formula for $\int \frac{\sin^m x}{\cos^n x}\, dx$ where $m, n$ are both positive integers

Solution: Let $I_{m,n} = \int \frac{\sin^m x}{\cos^n x}\, dx$

Now $$\frac{d}{dx}\left(\frac{\sin^m x}{\cos^n x}\right) = \frac{\cos^n x\,(m \sin^{m-1} x \cos x) - \sin^m x\,(n \cos^{n-1} x(-\sin x))}{(\cos^n x)^2}$$

$$= m\frac{\sin^{m-1} x}{\cos^{n-1} x} + n\frac{\sin^{m+1} x}{\cos^{n+1} x}$$

Integrating both sides, we get

$$\frac{\sin^m x}{\cos^n x} = m I_{m-1,n-1} + n I_{m+1,n+1}$$

Therefore, replacing $m$ by $m-1$ and $n$ by $n-1$, we get

$$(n-1) I_{m,n} + (m-1) I_{m-2,n-2} = \frac{\sin^{m-1} x}{\cos^{n-1} x}$$

or, $$I_{m,n} = \frac{1}{n-1}\frac{\sin^{m-1} x}{\cos^{n-1} x} - \frac{m-1}{n-1} I_{m-2,n-2}$$

5. Obtain a reduction formula for $\int \tan^n x\, dx$ where $n$ is a positive integer.

Solution: Let $I_n = \int \tan^n x$. Then $I_n = \int \tan^{n-2} x \tan^2 x\, dx$

$$= \int \tan^{n-2} x(\sec^2 x - 1)\, dx = \int \tan^{n-2} x \sec^2 x\, dx - I_{n-2}$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

So, a reduction formula is $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

6. Find a reduction formula for ~~fractn~~ $\int \sec^n x\,dx$ ($n$ being a positive integer).

Solution: Let $I_n = \int \sec^n x\,dx = \int \sec^{n-2}x \cdot \sec^2 x\,dx$

$$= \sec^{n-2}x\tan x - (n-2)\int \sec^{n-3}x\,\sec x\tan x\cdot\tan x\,dx$$

$$= \sec^{n-2}x\tan x - (n-2)\int \sec^{n-2}x\,(\sec^2 x-1)\,dx$$

$$= \sec^{n-2}x\tan x - (n-2)\int \sec^{n}x\,dx + (n-2)\int \sec^{n-2}x\,dx$$

$$= \sec^{n-2}x\tan x - (n-2)I_n + (n-2)I_{n-2}$$

So, $I_n = \dfrac{\sec^{n-2}x\tan x}{n-1} + \dfrac{n-2}{n-1}I_{n-2}$ which is a required reduction formula.

Exercises: 1. If $I_n = \int_0^{\pi/4} \tan^n x\,dx$, $n$ being a positive integer greater than 1, then show that $I_{n+1} + I_{n-1} = \frac{1}{n}$

2. Find a reduction formula for $\int e^{-x} x^n\,dx$

3. If $I_n = \int x^n \cos ax\,dx$ and $J_n = \int x^n \sin ax\,dx$, then show that

(i) $aI_n = x^n \sin ax - nJ_{n-1}$

(ii) $aJ_n = -x^n\cos ax + nI_{n-1}$

(iii) $a^2 I_n = x^{n-1}(ax\sin ax + n\cos ax) - n(n-1)I_{n-2}$

(iv) $a^2 J_n = x^{n-1}(n\sin ax \cancel{+} - ax\cos ax) - n(n-1)J_{n-2}$

4. Find a reduction formula for $\int e^{ax}\cos^n x\,dx$, $n$ is a positive integer greater than 1.

5. If $I_n = \int_0^{\pi/2} x^n \sin x\,dx$ and $n$ is a positive integer greater than 1 then show that $I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

Worked out exercises

1. If $I_n = \int (\sin x + \cos x)^n dx$, show that

$$nI_n = (\sin x + \cos x)^{n-1}(\sin x - \cos x) + 2(n-1)I_{n-2},$$ n is a positive integer greater than 1.

Solution: $I_n = \int (\sin x + \cos x)^n dx = \int (\sin x + \cos x)^{n-1} (\sin x + \cos x) dx$

$= (\sin x + \cos x)^{n-1}(\cancel{\sin}\ \sin x - \cos x) - (n-1)\int (\sin x + \cos x)^{n-2}(\cos x - \sin x)(\sin x - \cos x) dx$

$= (\sin x + \cos x)^{n-1}(\sin x - \cos x) + (n-1)\int (\sin x + \cos x)^{n-2}(\cos x - \sin x)^2 dx$

$= (\sin x + \cos x)^{n-1}(\sin x - \cos x) + (n-1)\int (\sin x + \cos x)^{n-2}\{(\cos x + \sin x)^2 - 2\sin 2x\} dx$

$= (\sin x + \cos x)^{n-1}(\sin x - \cos x) + (n-1)I_n - 2(n-1)\int (\sin x + \cos x)^{n-2}\{(\sin x + \cos x)^2 - 1\} dx$

$= (\sin x + \cos x)^{n-1}(\sin x - \cos x) + (n-1)I_n - 2(n-1)I_n + 2(n-1)I_{n-2}$

or, $\cancel{nI_n = (\sin x + \cos x)^{n-1}(\sin x - \cos x) + (n-1)I_n - 2(n-1)I_n + 2(n-1)I_{n-2}}$

or, $nI_n = (\sin x + \cos x)^{n-1}(\sin x - \cos x) + 2(n-1)I_{n-2}$

2. If $I_n = \int \frac{\sin(2n-1)x}{\sin x} dx$ and $J_n = \int \frac{\sin^2 nx}{\sin^2 x} dx$, ($n$ is a positive integer)

show that (i) $n(I_{n+1} - I_n) = \sin 2nx$

(ii) $J_{n+1} - J_n = I_{n+1}$

Solution: $I_n = \int \frac{\sin(2n-1)x}{\sin x} dx$ and $J_n = \int \frac{\sin^2 nx}{\sin^2 x} dx$

(i) $I_{n+1} - I_n = \int \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x} dx = \int \frac{2\cos 2nx \sin x}{\sin x} dx$

$= \int 2\cos 2nx\, dx = \frac{2\sin 2nx}{2n} = \frac{\sin 2nx}{n}$

So, $n(I_{n+1} - I_n) = \sin 2nx$

(ii) $J_{n+1} - J_n = \int \frac{\sin^2(n+1)x - \sin^2 nx}{\sin^2 x} dx$

$= \int \frac{\sin(2n+1)x \sin x}{\sin^2 x} dx = \int \frac{\sin(2n+1)x}{\sin x} dx = I_{n+1}$