8. Show that $\int_0^\infty e^{-ax}\cos bx = \frac{a}{a^2+b^2}$; $a>0$

Solution: $\int e^{-ax}\cos bx = -\frac{e^{-ax}}{a}\cos bx - \frac{b}{a}\int e^{-ax}\sin bx\,dx$ (Integrating by parts)

$$= -\frac{e^{ax}\cos bx}{a} - \frac{b}{a}\left[-\frac{e^{ax}\sin bx}{a} + \frac{b}{a}\int e^{-ax}\cos bx\,dx\right]$$

$$\therefore \quad \frac{a^2+b^2}{a^2}\int e^{-ax}\cos bx = -\frac{e^{-ax}\cos bx}{a} + \frac{b}{a^2}e^{-ax}\sin bx$$

So, $$\lim_{R\to\infty}\int_0^R e^{-ax}\cos bx = \frac{a^2}{a^2+b^2}\lim_{R\to\infty}\left[\frac{e^{-ax}}{a^2}\left(\cancel{a\cos bx}\; b\sin bx - a\cos bx\right)\right]_0^R$$

$$= \frac{1}{a^2+b^2}\lim_{R\to\infty}\left[e^{-aR}\left(b\sin bR - a\cos bR\right)+a\right]$$

$$= \frac{a}{a^2+b^2}$$

So, $$\int_0^\infty e^{-ax}\cos bx = \frac{a}{a^2+b^2}$$

Exercise 9. Show that $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$; $a, b>0$

10. Show that $\int_0^\infty \frac{x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\log\left(\frac{a}{b}\right)$; $a, b>0$

11. Show that $\int_0^\infty e^{-ax}\sin bx = \frac{b}{a^2+b^2}$; $a>0$

Worked out exercises: 12. Show that the following integrals converge: (i) $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$ (ii) $\int_0^{1/2} \frac{dx}{x^{1/2}(1-x)^{1/3}}$

(iii) $\int_0^\infty \frac{dx}{e^x+1}$

(iv) $\int_0^\infty \frac{x^2}{\sqrt{x^7+1}}\,dx$

Solutions: ~~(3)~~(i) Here the improper integral is $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$. Here $x=0$ is the point of infinite discontinuity.

Now $\lim_{x\to 0+} (x-0)^{\frac{1}{2}} \cdot \frac{1}{(1+x)\,x^{1/2}} = \lim_{x\to 0+} \frac{x^{1/2}}{(1+x)\,x^{1/2}} = 1 \neq 0$

Here $\mu = \frac{1}{2} < 1$. Hence the integral $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$ converges.

(ii) Here the integral is $\int_0^{1/2} \frac{dx}{x^{1/2}(1-x)^{1/3}}$. Now

$\lim_{x\to 0+} (x-0)^{1/2} \frac{1}{x^{1/2}(1-x)^{1/3}} = \lim_{x\to 0+} \frac{x^{1/2}}{x^{1/2}(1-x)^{1/3}} = 1 \neq 0$

Here $\mu = \frac{1}{2} < 1$. Hence the integral is convergent

(iii) Here the integral is $\int_0^\infty \frac{dx}{e^x+1}$

Now $0 \le \frac{1}{e^x+1} < \frac{1}{e^x} = e^{-x}$ and $\int_0^\infty e^{-x}\,dx$ converges

as $\lim_{R\to\infty} \int_0^R e^{-x}\,dx = \lim_{R\to\infty} \left[-e^{-x}\right]_0^R = \lim_{R\to\infty} \left(-e^{-R}+1\right) = 1$

So, by comparison test $\int_0^\infty \frac{dx}{e^x+1}$ converges.

~~(iv) Here the integral is~~ $\cancel{\int_0^\infty \frac{\sin x\,dx}{1+\cos x+e^x}}$

Now $\cancel{\lim_{x\to\infty} (x-0)^2 \frac{\sin x}{1+\cos x+e^x} =}$

(iv) Here the integral is $\int_0^\infty \frac{x^2}{\sqrt{x^7+1}}\,dx$

Now $\lim_{x\to\infty} x^{3/2} \cdot \frac{x^2}{\sqrt{x^7+1}} = \lim_{x\to\infty} \frac{x^{7/2}}{x^{7/2}\sqrt{1+\frac{1}{x^7}}} = 1 \neq 0$

Here $\mu = \frac{3}{2} > 1$. So the integral is convergent

Exercises. 13. Show that the following integrals converge : (i) $\int_0^1 \frac{dx}{(x^2+x)^{1/3}}$ (ii) $\int_0^\infty \frac{\cos x}{1+x^2}\,dx$

(iii) $\int_1^\infty \frac{(x-1)\sqrt{x}}{1+x+x^3+\ln x}\,dx$

Worked out exercises

14. Discuss the convergence of the integral $\int_0^\infty e^{-x}x^{n-1}\,dx$

Solution: Let $I_1 = \int_0^1 e^{-x}x^{n-1}\,dx$ and $I_2 = \int_1^\infty e^{-x}x^{n-1}\,dx$

The part I_1 is proper when $n \geq 1$, improper when $n<1$

Now $\lim\limits_{x\to 0+} x^{1-n} f(x) = \lim\limits_{x\to 0+} x^{1-n}.e^{-x}.x^{n-1} = \lim\limits_{x\to 0+} e^{-x} = 1 \neq 0$

So, I_1 converges when $1-n<1$ or $n>0$, by μ-test.

Now for $n>0$, the part I_2 converges as

$$\lim_{x\to\infty} x^2.e^{-x}x^{n-1} = \lim_{x\to\infty} e^{-x}x^{n+1} = 0$$

So the integral $\int_0^\infty e^{-x}x^{n-1}\,dx$ converges when $n>0$

15. Show that the integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ converges if and only if $m>0,\ n>0$

Solution: This integral is proper when $m>1$, $n>1$ but is improper at the lower limit where $m<1$ and at the upper limit when $n<1$. We therefore split the integral into two parts I_1 and I_2 where

$$I_1 = \int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx \quad \text{and} \quad I_2 = \int_{1/2}^1 x^{m-1}(1-x)^{n-1}\,dx.$$

Let $f(x) = x^{m-1}(1-x)^{n-1}$. We now first check the convergence of I_1. Here the integrand is positive for $x \in (0, \frac{1}{2}]$. $\lim_{x \to 0+} x^{1-m} f(x) = \lim_{x \to 0+} (1-x)^{n-1} = 1 \neq 0$

So, by μ-test I_1 is convergent if $1-m < 1$ or $m > 0$

~~Conv~~ Now we test the convergence of $I_2 = \int_{1/2}^{1} x^{m-1}(1-x)^{n-1} dx$

Here $f(x) > 0$ for $x \in [\frac{1}{2}, 1)$.

$\lim_{x \to 1-} (1-x)^{1-n} f(x) = \lim_{x \to 1-} (1-x)^{1-n} x^{m-1} (1-x)^{n-1}$

$= \lim_{x \to 1-} x^{m-1} = 1$. So the integral I_2 is convergent by μ-test if $1-n < 1$ or, $n > 0$

As $I = I_1 + I_2$. So, the integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ is convergent if and only if $m > 0$ and $n > 0$

Beta function and Gamma function

The improper integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ is convergent if $m > 0, n > 0$. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}$, $m > 0, n > 0$ is called the Beta function and it is denoted by $B(m,n)$

So, $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$, $m > 0$, $n > 0$

Properties

1. $B(1,1) = 1$

Proof: $B(1,1) = \int_0^1 dx = 1$

2. $B(m,n) = B(n,m)$

Proof: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$, $m > 0$, $n > 0$

$= \lim_{\substack{\varepsilon \to 0+ \\ \delta \to 0+}} \int_{\varepsilon}^{1-\delta} x^{m-1}(1-x)^{n-1} dx$