Improper Integrals

The theory of integration was developed under two assumptions:

~~(i) the integrand was required to be bounded on the~~

(i) the interval of integration was required to be a closed and bounded interval, and

(ii) the integrand was required to be bounded on the interval.

In short, the theory for the integral $\int_a^b f(x)\,dx$ was developed when $[a, b]$ is a closed an bounded interval and f is bounded on $[a, b]$.

If these two restrictions are relaxed, we have the following two types of integrals, called improper integrals —

(a) ~~impro~~ improper integrals on an unbounded ~~integral~~ interval.

(b) improper integrals on a finite interval when the integrand is unbounded.

A. Improper integrals on an unbounded interval

I. Improper integral $\int_a^\infty f(x)\,dx$ where $f(x)$ is integrable on $[a, X]$ for all $X > a$

$$\text{Let } \phi(X) = \int_a^X f(x)\,dx, \quad X > a$$

If $\lim_{X\to\infty} \phi(X)$ exits (finitely) then the improper integral $\int_a^\infty f(x)\,dx$ is said to be convergent. If the limit be l, we write $\int_a^\infty f(x)\,dx = l$.

If $\lim_{X\to\infty} \phi(X)$ does not exists (finitely) then the improper integral $\int_a^\infty f(x)\,dx$ is said to be divergent

Examples 1. Consider the integral $\int_0^\infty e^{-x}dx$. The integrand is integrable on any closed interval $[0, X]$, $X > 0$. The integral is improper. Let $\phi(X) = \int_0^X e^{-x}dx$, $X > 0$

Then $\phi(X) = \left[-e^{-x}\right]_0^X = 1 - e^{-X}$. So, $\lim_{X\to\infty}(1-e^{-X}) = 1$

So, $\int_0^\infty e^{-x}dx$ is convergent and $\int_0^\infty e^{-x}dx = 1$

2. Consider the integral $\int_0^\infty \frac{1}{1+x}dx$. The integrand is integrable on any closed interval $[0, X]$, $X > 0$. The integral is improper.

Let $\phi(X) = \int_0^X \frac{1}{1+x}dx$, $X > 0$ Then $\phi(X) = \left[\log(1+x)\right]_0^X$ $= \log(1+X)$. So $\lim_{X\to\infty}\phi(X) = \infty$. So, the integral $\int_0^\infty \frac{1}{1+x}dx$ is divergent.

II. Convergence of the improper integral $\int_{-\infty}^b f(x)dx$ where f is integrable on $[X, b]$ for all $X < b$.

Let $\phi(X) = \int_X^b f(x)dx$, $X < b$

If $\lim_{X\to-\infty}\phi(X)$ exists (finitely) then the improper integral $\int_{-\infty}^b f(x)dx$ is said to be convergent. If the limit be l, we write $\int_{-\infty}^b f(x)dx = l$

Example : 1. Consider the integral $\int_{-\infty}^0 e^x dx$

$\phi(X) = \int_X^0 e^x dx = \left[e^x\right]_X^0 = (1 - e^X)$, $X < 0$

$\therefore \lim_{X \to -\infty} \phi(X) = \lim_{X \to -\infty} (1 - e^{X}) = 1$

So, $\int_{-\infty}^{0} e^{x} dx$ is convergent and $\int_{-\infty}^{0} e^{x} dx = 1$

2. Consider $\int_{-\infty}^{0} \frac{1}{1-x} dx$

$$\phi(X) = \int_{X}^{0} \frac{1}{1-x} dx = \left[-\log(1-x)\right]_{X}^{0} = \log(1-X), \quad X < 0$$

So, $\lim_{X \to -\infty} \phi(X) = \lim_{X \to -\infty} \log(1-X) = \infty$

So, $\int_{-\infty}^{0} \frac{1}{1-x} dx$ diverges.

$\underline{\text{III}}$ Convergence of the improper integral $\int_{-\infty}^{\infty} f(x) dx$ where f is integrable on $[X_1, X_2]$ for all $X_1, X_2 \in \mathbb{R}$ (the set of all real numbers) satisfying $X_1 < X_2$.

Let $c \in \mathbb{R}$. If both the integrals $\int_{-\infty}^{c} f(x) dx$ and $\int_{c}^{\infty} f(x) dx$ be convergent according to definition I and II above, then the improper integral $\int_{-\infty}^{\infty} f(x) dx$ is said to be convergent and we write,

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{c} f(x) dx + \int_{c}^{\infty} f(x) dx.$$

Example: consider the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} dx$

Here we consider $\int_{-\infty}^{a} \frac{dx}{1+x^2}$ and $\int_{a}^{\infty} \frac{dx}{1+x^2}$ where $a \in \mathbb{R}$

Let $\phi(X) = \int_{X}^{a} \frac{dx}{1+x^2}$, $X < a$ Then $\phi(X) = \left[\tan^{-1} x\right]_{X}^{a} = \tan^{-1} a - \tan^{-1} X$

So, $\lim_{X \to -\infty} \left[\tan^{-1} a - \tan^{-1} X\right] = \tan^{-1} a + \pi/2$

So, the improper integral $\int_{-\infty}^{a} \frac{dx}{1+x^2}$ is convergent.

Let $\psi(X) = \int_a^X \frac{dx}{1+x^2}$, $X > a$ Then $\psi(X) = \left[\tan^{-1} x\right]_a^X = \tan^{-1} X - \tan^{-1} a$

So, $\lim_{X\to\infty} \left[\tan^{-1} X - \tan^{-1} a\right] = \pi/2 - \tan^{-1} a$

So, the improper integral $\int_a^{\infty} \frac{dx}{1+x^2}$ is convergent.

So, the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is convergent and

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \left(\tan^{-1} a + \pi/2\right) + \left(\pi/2 - \tan^{-1} a\right) = \pi$$

B. Improper integrals on a closed and bounded interval, the integrand having infinite discontinuities.

I Convergence of the improper integral $\int_a^b f(x)dx$ when a is the only point of infinite discontinuity. Let a be the only point of infinite discontinuity of a function f which is bounded and integrable in $[a+\varepsilon, b]$ for every ε satisfying $0 < \varepsilon < b-a$

Let $\phi(\varepsilon) = \int_{a+\varepsilon}^{b} f(x)\,dx$, $0 < \varepsilon < b-a$

If $\lim_{\varepsilon\to 0+} \phi(\varepsilon)$ exists (finitely) then the improper integral $\int_a^b f(x)dx$ is said to be convergent. If the limit be l, we write $\int_a^b f(x)dx = l$.

If $\lim_{\varepsilon\to 0+} \phi(\varepsilon)$ does not exist (finitely) then the improper integral $\int_a^b f(x)dx$ is said to be divergent.