Examples 1. The integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is improper, since 0 is only point of infinite discontinuity of the integrand. The integrand is bounded and integrable in $[0+\varepsilon, 1]$ for all ε satisfying $0<\varepsilon<1$

$$\lim_{\varepsilon\to 0+} \int_{0+\varepsilon}^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0+}\left[2\sqrt{x}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0+}\left[2-2\sqrt{\varepsilon}\right] = 2$$

So, the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent and $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$

2. The integral $\int_0^1 \frac{1}{x}\,dx$ is improper, since 0 is only point of infinite discontinuity for the integrand. The integrand is bounded and integrable on $[0+\varepsilon, 1]$ for all ε satisfying $0<\varepsilon<1$.

$$\lim_{\varepsilon\to 0+} \int_{0+\varepsilon}^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0+}\left[\log x\right]_\varepsilon^1 = \lim_{\varepsilon\to 0}\left(-\log\varepsilon\right) = \infty$$

So the improper integral $\int_0^1 \frac{dx}{x}$ is divergent.

II. Convergence of the improper integral $\int_a^b f(x)\,dx$ when b is the only point of infinite discontinuity of f in $[a,b]$.

Let the right end point b of the closed and bounded interval $[a,b]$ be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a, b-\varepsilon]$ for every ε satisfying $0<\varepsilon<b-a$.

Let $\phi(\varepsilon) = \int_0^{b-\varepsilon} f(x)\,dx$, $0<\varepsilon<b-a$

If $\lim_{\varepsilon\to 0+}\phi(\varepsilon)$ exists (finitely) then the improper integral $\int_a^b f(x)\,dx$ is said to be convergent. If the limit is l, we write $\int_a^b f(x)\,dx = l$.

If $\lim_{\varepsilon\to 0+}\phi(\varepsilon)$ does not exist (finitely) then the improper integral $\int_a^b$ is said to be divergent.

Examples 1. The integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ is improper, since 1 is only point of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0, 1-\varepsilon]$ for all ε satisfying $0<\varepsilon<1$.

$$\lim_{\varepsilon\to 0+}\int_0^{1-\varepsilon}\frac{1}{\sqrt{1-x^2}}dx = \lim_{\varepsilon\to 0+}\left[\sin^{-1}x\right]_0^{1-\varepsilon} = \lim_{\varepsilon\to 0+}\left(\sin^{-1}(1-\varepsilon)\right) = \pi/2$$

Therefore the ~~integral~~ improper integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ is convergent and $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \pi/2$

2. The integral $\int_0^1 \frac{1}{1-x}dx$ is improper, since 1 is ~~the~~ only point of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0, 1-\varepsilon]$ for all ε satisfying $0<\varepsilon<1$

$$\lim_{\varepsilon\to 0+}\int_0^{1-\varepsilon}\frac{1}{1-x}dx = \lim_{\varepsilon\to 0+}\left[-\log(1-x)\right]_0^{1-\varepsilon}$$

$$= \lim_{\varepsilon\to 0+}\left[-\log\varepsilon\right] = \infty$$

Therefore the ~~int~~ improper integral $\int_0^1\frac{1}{1-x}dx$ is divergent.

III convergence of the improper integral $\int_a^b f(x)dx$ when a and b are the only points of infinite discontinuity of f in $[a,b]$

Let the end points a, b of the closed and bounded interval $[a,b]$ be the only points of infinite discontinuity of ~~f~~ a ~~in [a,b] the~~ function f which is bounded and integrable on $[a+\varepsilon, b-\varepsilon']$ for every $\varepsilon, \varepsilon'$ satisfying $0<\varepsilon<b-a$, $0<\varepsilon'<b-a$

Let ~~x~~ $c \in (a,b)$ i.e., $a < c < b$

If the improper integral $\int_a^c f(x)\,dx$ & $\int_c^b f(x)\,dx$ be both convergent according ~~the~~ to the definition given above, Then the improper integral $\int_a^b f(x)\,dx$ is said to be convergent and we write $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Examples: 1. The integral $\int_0^2 \frac{1}{\sqrt{x(2-x)}}$ is improper, since 0 and 2 are only points of infinite discontinuity of the integrand. The integrand is bounded and integrable on $[0+\varepsilon, 2-\varepsilon']$ for all $\varepsilon, \varepsilon'$ satisfying $0 < \varepsilon < 2$, $0 < \varepsilon' < 2$

Let us examine $\lim_{\varepsilon \to 0+} \int_\varepsilon^1 \frac{1}{\sqrt{x(2-x)}}\,dx$ and $\lim_{\varepsilon' \to 0+} \int_1^{2-\varepsilon'} \frac{1}{\sqrt{x(2-x)}}\,dx$

Now $\lim_{\varepsilon \to 0+} \int_\varepsilon^1 \frac{1}{\sqrt{x(2-x)}}\,dx = \lim_{\varepsilon \to 0+} \left[\sin^{-1}(x-1)\right]_\varepsilon^1 = \frac{\pi}{2}$

and $\lim_{\varepsilon' \to 0+} \int_1^{2-\varepsilon'} \frac{1}{\sqrt{x(2-x)}} = \lim_{\varepsilon' \to 0+} \left[\sin^{-1}(x-1)\right]_1^{2-\varepsilon'} = \frac{\pi}{2}$

So, $\int_0^2 \frac{1}{\sqrt{x(2-x)}}\,dx$ is convergent and

$$\int_0^2 \frac{1}{\sqrt{x(2-x)}}\,dx = \frac{\pi}{2} + \pi/2 = \pi$$

<u>IV</u> Convergence of the improper integral $\int_a^b f(x)\,dx$ when there only one point of discontinuity of the integrand at c, where $a < c < b$.

If the improper integral $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ be both convergent according to the definition given above, then the improper integral $\int_a^b f(x)dx$ is said to be convergent and we write $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

So, if both limits $\lim_{\varepsilon\to 0+} \int_a^{c-\varepsilon} f(x)dx$ and $\lim_{\varepsilon'\to 0+} \int_{c+\varepsilon'}^b f(x)dx$ exist then $\int_a^b f(x)dx = \lim_{\varepsilon\to 0+} \int_a^{c-\varepsilon} f(x)dx + \lim_{\varepsilon'\to 0} \int_{c+\varepsilon'}^b f(x)dx$

If the improper integral $\int_a^b f(x)dx$ is convergent, its value is equal to the symmetric limit $\left[\lim_{\varepsilon\to 0+} \int_a^{c-\varepsilon} f(x)dx + \lim_{\varepsilon\to 0+} \int_{c+\varepsilon}^b f(x)dx\right]$

It may happen that the improper integral $\int_a^b f(x)dx$ is divergent but the limit $\lim_{\varepsilon\to 0} \left[\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx\right]$ exists, then this symmetric limit is called the Cauchy principal value of the improper integral integral $\int_a^b f(x)dx$ and it is denoted by $P\int_a^b f(x)dx$

For example, let us consider the improper integral $\int_{-1}^1 f(x)dx$ where

$$f(x) = \frac{1}{x}, \ x \neq 0$$
$$= 0, \ x = 0$$

Hence $\lim_{\varepsilon\to 0+} \int_{-1}^{0-\varepsilon} f(x)dx + \lim_{\varepsilon'\to 0+} \int_{0+\varepsilon'}^1 f(x)dx$

$= \lim_{\varepsilon\to 0+} \log\varepsilon + \lim_{\varepsilon'\to 0+} (-\log\varepsilon')$ as this limit does not exists if $\varepsilon\to 0$, $\varepsilon'\to 0$ independently

But $\lim_{\varepsilon\to 0+} \left[\int_{-1}^{0-\varepsilon} f(x)dx + \int_{0+\varepsilon}^1 f(x)dx\right] = \lim_{\varepsilon\to 0+} (\log\varepsilon - \log\varepsilon) = 0$

So, $\int_{-1}^1 f(x)dx$ is divergent but $P\int_{-1}^1 f(x)dx = 0$