V Convergence of the improper integral $\int_a^b f(x)dx$ when a finite number of points $c_1, c_2, \ldots, c_m$ are the only points of infinite discontinuity of f in $[a,b]$.

Case 1 Let $a < c_1 < c_2 < \cdots < c_m < b$.

If the integrals $\int_a^{c_1} f(x)dx, \int_{c_1}^{c_2} f(x)dx, \ldots, \int_{c_m}^{b} f(x)dx$, ~~all~~ be exist all convergent according to the definition given above, then the improper integral $\int_a^b f(x)dx$ is said to be convergent and we write

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_m}^{b} f(x)dx$$

Case 2 Either $a = c_1$ or $b = c_m$ or both

If $a = c_1$, then $\int_a^b f(x)dx = \int_a^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx + \cdots + \int_{c_m}^{b} f(x)dx$, provided each integral in the right hand side is convergent according to the definition given above

If $b = c_m$, then $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_{m-1}}^{b} f(x)dx$, provided each integral in the right hand side ~~is~~ is convergent according to the definition given above

If $a = c_1$ and $b = c_m$ then $\int_a^b f(x)dx = \int_{a}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx + \cdots + \int_{c_{m-1}}^{b} f(x)dx$, provided each integral in the right hand side is convergent according to the definition given above.

Statement of Comparison test

Let a ~~and~~ be the only point of infinite discontinuity of the functions f and g which are both integrable on $[a+\varepsilon, b]$ for all ε satisfying $0 < \varepsilon < b-a$ and $0 < f(x) \le k\,g(x)$ for all $x \in (a, b]$, where $k > 0$ Then we have,

(i) $\int_a^b g(x)\,dx$ is convergent $\Rightarrow$ $\int_a^b f(x)\,dx$ is convergent;

(ii) $\int_a^b f(x)\,dx$ is divergent $\Rightarrow$ $\int_a^b g(x)\,dx$ is divergent.

~~Assume~~ A useful comparison integral for $[a, b]$

The integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $\mu < 1$

Proof: The integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is proper if $\mu \leq 0$

Let $\mu > 0$. Let $f(x) = \frac{1}{(x-a)^\mu}$, $a < x \leq b$. a is the only point of infinite discontinuity of f. f is integrable in $[a+\varepsilon, b]$ for $0 < \varepsilon < b-a$ and $f(x) > 0$ for all $x \in (a, b]$

Let $\phi(\varepsilon) = \int_{a+\varepsilon}^b \frac{dx}{(x-a)^\mu}$, $0 < \varepsilon < b-a$

If $\mu \neq 1$, $\phi(\varepsilon) = \int_{a+\varepsilon}^b (x-a)^{-\mu} dx = \frac{1}{1-\mu}\left[\frac{1}{(b-a)^{\mu-1}} - \frac{1}{\varepsilon^{\mu-1}}\right]$

If $0<\mu<1$, $\lim_{\varepsilon \to 0+} \phi(\varepsilon) = \frac{1}{1-\mu} \cdot \frac{1}{(b-a)^{\mu-1}}$ and

if $\mu > 0$ $\lim_{\varepsilon \to 0+} \phi(\varepsilon) = \infty$

If $\mu = 1$, $\phi(\varepsilon) = \int_{a+\varepsilon}^b \frac{dx}{x-a} = \log|b-a| - \log|\varepsilon|$ and

$\lim_{\varepsilon \to 0+} \phi(\varepsilon) = \infty$

Since $\lim_{\varepsilon \to 0+} \phi(\varepsilon)$ exists finitely when $0<\mu<1$ and $\lim_{\varepsilon \to 0+} \phi(\varepsilon) = \infty$ when $\mu \geq 1$, the improper integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $0<\mu<1$.

Since $\int_a^b \frac{dx}{(x-a)^\mu}$ is proper if $\mu \leq 0$ and the improper integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if

$0<\mu<1$, it follows that the integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $\mu<1$

Statement of μ-test for $[a,b]$

Let a be the only point of ~~discontinuity~~ infinite discontinuity of a function f which is integrable ~~in~~ on $[a+\varepsilon, b]$ for $0<\varepsilon<b-a$ and $f(x)>0$ for all ~~$x\in[a,b]$~~ $x\in(a,b]$.

If $\lim_{x\to a+} (x-a)^\mu f(x) = l$ where l is a non-zero finite number, then the integral $\int_a^b f(x)\,dx$ is convergent if and only if $\mu<1$.

Statement of comparison test for unbounded interval.

Let the functions f and g be both integrable on $[a, X]$ for all $X>a$ and $0<f(x)\le k g(x)$ for all $x\geq a$, where $k>0$.

Then (i) $\int_a^\infty g(x)\,dx$ is convergent $\Rightarrow \int_a^\infty f(x)\,dx$ is convergent

(ii) $\int_a^\infty f(x)\,dx$ is divergent $\Rightarrow \int_a^\infty g(x)\,dx$ is divergent

A useful comparison integral for unbounded interval.

The improper integral $\int_a^\infty \frac{dx}{x^\mu}$, where $a>0$, is convergent if and only $\mu>1$

Proof. Let $\phi(X) = \int_a^X \frac{dx}{x^\mu}$, $X>a$.

If $\mu\neq 1$, ~~so~~ we have $\phi(X) = \int_a^X \frac{dx}{x^\mu} = \left[\frac{x^{1-\mu}}{1-\mu}\right]_a^X = \frac{1}{1-\mu}\left[X^{1-\mu} - a^{1-\mu}\right]$

and if $\mu=1$, $\phi(X) = \int_a^X \frac{dx}{x} = \log|X| - \log|a|$ $(a>0)$

If $\mu=1$, $\lim_{X\to\infty}\phi(X) = \lim_{X\to\infty}\left[\log|X| - \log|a|\right] = \infty$

If $\mu\neq 1$, $\lim_{X\to\infty}\phi(X) = \lim_{X\to\infty}\frac{1}{1-\mu}\left[X^{1-\mu} - a^{1-\mu}\right]$

So, if $\mu \neq 1$ $\lim\limits_{X\to\infty} \phi(X) = \infty$ if $\mu < 1$

$$= \frac{1}{(\mu-1)\,a^{\mu-1}}, \text{ if } \mu > 1$$

Since $\lim\limits_{X\to\infty} \phi(X)$ exists finitely when $\mu > 1$ and $\lim\limits_{X\to\infty} \phi(X) = \infty$ when $\mu \leq 1$, the improper integral $\int_a^\infty \frac{dx}{x^\mu}$ is convergent if $\mu > 1$ and divergent if $\mu \geq 1$

Hence the improper integral $\int_a^\infty \frac{dx}{x^\mu}$, where $a > 0$, is convergent if and only if $\mu > 1$

Statement of μ-test for unbounded interval

Let $f(x) > 0$ for all $x \geq a$. If $\lim\limits_{x\to\infty} x^\mu f(x) = l$, where l is a non-zero finite number, the improper integral $\int_a^\infty f(x)\,dx$ is convergent if and only if $\mu > 1$

Worked out exercises

1. Examine whether the following improper integrals exists or not. Find the value if one converges:

(a) $\int_0^\infty e^{-x}\,dx$ (b) $\int_0^\infty \frac{dx}{4+9x^2}$

(c) $\int_1^\infty \frac{dx}{x(1+x)}$ (d) $\int_0^\infty \frac{x\,dx}{x^2+4}$

Solution: (a) $\lim\limits_{X\to\infty}\int_0^X e^{-x}\,dx = \lim\limits_{X\to\infty}\left[-e^{-x}\right]_0^X = \lim\limits_{X\to\infty}\left(1 - e^{-X}\right) = 1$

So, $\int_0^\infty e^{-x}\,dx$ is convergent and $\int_0^\infty e^{-x}\,dx = 1$

(b) $\lim\limits_{X\to\infty}\int_0^X \frac{dx}{4+9x^2} = \frac{1}{9}\lim\limits_{X\to\infty}\int_0^X \frac{dx}{x^2+\left(\frac{2}{3}\right)^2} = \frac{1}{9}\lim\limits_{X\to\infty}\frac{3}{2}\left[\tan^{-1}\frac{3x}{2}\right]_0^X$

$= \frac{1}{6}\lim\limits_{X\to\infty}\left[\tan^{-1}\frac{3X}{2}\right] = \frac{1}{6}\times\frac{\pi}{2} = \frac{\pi}{12}$. So $\int_0^\infty \frac{dx}{4+9x^2}$ is convergent and $\int_0^\infty \frac{dx}{4+9x^2} = \frac{\pi}{12}$