(c) $\lim\limits_{R\to\infty}\int_1^R \frac{dx}{x(1+x)} = \lim\limits_{R\to\infty}\int_1^R\left(\frac{1}{x}-\frac{1}{1+x}\right)dx = \lim\limits_{R\to\infty}\left[\log\left|\frac{x}{1+x}\right|\right]_1^R$

$= \lim\limits_{R\to\infty}\left[\log\frac{R}{1+R} - \log\frac{1}{2}\right] = 0 - \log\frac{1}{2} = \log 2$

So, $\int_1^\infty \frac{dx}{x(1+x)}$ is convergent and $\int_1^\infty \frac{dx}{x(1+x)} = \log 2$.

(d) $\lim\limits_{M\to\infty}\int_0^M \frac{x\,dx}{x^2+4} = \frac{1}{2}\lim\limits_{M\to\infty}\int_0^M \frac{2x\,dx}{x^2+4} = \frac{1}{2}\lim\limits_{M\to\infty}\left[\log(x^2+4)\right]_0^M$

$= \frac{1}{2}\lim\limits_{M\to\infty}\left[\log(M^2+4) - \log 4\right] = \infty$

Hence $\int_0^\infty \frac{x\,dx}{x^2+4}$ is divergent.

Exercise 2 Examine whether the following improper integrals exits or not. Find the value if one converges:

(a) $\int_1^\infty \frac{dx}{x^{3/2}}$ (b) $\int_{-\infty}^0 e^{2x}dx$ (c) $\int_{-\infty}^\infty \frac{x\,dx}{x^2+4}$

(d) $\int_0^\infty \frac{dx}{(x+1)(x+2)}$

Worked out Exercises

3. Examine whether the following improper integrals are convergent or not. Find the value if one converges:

(a) $\int_0^1 \frac{dx}{x^{2/3}}$ (b) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ (c) $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$

Solutions: (a) $\lim\limits_{\varepsilon\to 0+}\int_\varepsilon^1 \frac{dx}{x^{2/3}} = 3\lim\limits_{\varepsilon\to 0+}\left[x^{1/3}\right]_\varepsilon^1 = 3\lim\limits_{\varepsilon\to 0+}(1-\varepsilon^{1/3}) = 3$

So, $\int_0^1 \frac{dx}{x^{2/3}}$ is convergent and $\int_0^1 \frac{dx}{x^{2/3}} = 3$

(b) $\lim\limits_{\varepsilon\to 0+}\int_0^{3-\varepsilon} \frac{dx}{\sqrt{9-x^2}} = \lim\limits_{\varepsilon\to 0+}\left[\sin^{-1}\frac{x}{3}\right]_0^{3-\varepsilon} = \lim\limits_{\varepsilon\to 0+}\left[\sin^{-1}\frac{3-\varepsilon}{3}\right] = \pi/2$

So, $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ converges and $\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \pi/2$

(c) Let $I = \lim_{\varepsilon \to 0+} \int_\varepsilon^{1/2} \frac{dx}{x^{1/2}(1-x)^{1/2}} + \lim_{\varepsilon' \to 0} \int_{1/2}^{1-\varepsilon'} \frac{dx}{x^{1/2}(1-x)^{1/2}}$

Let $x = \sin^2\theta$, so, $dx = 2\sin\theta\cos\theta\, d\theta$

So, $\int \frac{dx}{x^{1/2}(1-x)^{1/2}} = \int \frac{2\sin\theta\cos\theta\, d\theta}{\sin\theta\cos\theta} = 2\theta = 2\sin^{-1}\sqrt{x}$

So, $I = \lim_{\varepsilon \to 0+}\left[2\sin^{-1}\tfrac{1}{2} - 2\sin^{-1}\sqrt{\varepsilon}\right] + \lim_{\varepsilon' \to 0}\left[2\sin^{-1}(1-\varepsilon') - 2\sin^{-1}\tfrac{1}{2}\right]$

$= 2\sin^{-1}\tfrac{1}{2} + \pi - 2\sin^{-1}\tfrac{1}{2} = \pi$

So, $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$ is convergent and $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}} = \pi$

Exercise 4 Examine whether the following integrals are convergent or not. Find the value if one converges.

(a) $\int_0^1 \frac{dx}{x^{1/2}}$ (b) $\int_0^1 \frac{dx}{1-x}$ (c) $\int_0^2 \frac{dx}{x^{1/3}}$

Worked out exercise : 5. Examine whether $\int_{-1}^1 \frac{dx}{x^3}$ exists or not. Find $P\int_{-1}^1 \frac{dx}{x^3}$ (i.e., the Cauchy principal value of $\int_{-1}^1 \frac{dx}{x^3}$)

Solution : $\lim_{\varepsilon \to 0+} \int_{-1}^{-\varepsilon} \frac{dx}{x^3} + \lim_{\varepsilon' \to 0+} \int_{\varepsilon'}^1 \frac{dx}{x^3}$

$= -\frac{1}{2}\lim_{\varepsilon \to 0+}\left[\frac{1}{\varepsilon^2} - 1\right] - \frac{1}{2}\lim_{\varepsilon' \to 0+}\left[1 - \frac{1}{\varepsilon'^2}\right]$

which does not exist.

Hence the integral is ~~divergent~~ not convergent.

But $P\int_{-1}^{1}\frac{dx}{x^3} = \lim_{\varepsilon\to 0+}\left[\int_{-1}^{0-\varepsilon}\frac{dx}{x^3} + \int_{\varepsilon}^{1}\frac{dx}{x^3}\right]$

$$= \lim_{\varepsilon\to 0+}\left\{-\frac{1}{2}\left[\frac{1}{x^2}\right]_{-1}^{-\varepsilon} - \frac{1}{2}\left[\frac{1}{x^2}\right]_{\varepsilon}^{1}\right\}$$

$$= -\frac{1}{2}\lim_{\varepsilon\to 0+}\left\{\frac{1}{\varepsilon^2} - 1 + 1 - \frac{1}{\varepsilon^2}\right\} = 0$$

which is Cauchy principal value.

Worked out exercises: 5. Show that

(a) $\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$ (b) $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$

Solutions: (a) $\lim_{X\to\infty}\int_0^X \frac{1}{1+x^2}\,dx = \lim_{X\to\infty}\left[\tan^{-1}X\right]_0^X$

$$= \lim_{X\to\infty}\left[\tan^{-1}X\right] = \pi/2$$

So, $\int_0^\infty \frac{1}{1+x^2}\,dx = \pi/2$

(b) $\lim_{\varepsilon\to 0+}\int_0^{1-\varepsilon}\frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon\to 0+}\left[\sin^{-1}x\right]_0^{1-\varepsilon}$

$$= \lim_{\varepsilon\to 0+}\left[\sin^{-1}(1-\varepsilon)\right] = \frac{\pi}{2}$$

Exercises 6. Show that

(a) $\int_{-\infty}^{\infty} x e^{-x^2}\,dx = 0$ (b) $\int_0^\infty \frac{x^2}{(1+x^2)^2}\,dx = \pi/4$

Worked out exercises:

7. Show that $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a+b)}$; $a, b > 0$

Solution: $\displaystyle\lim_{X\to 0}\int_0^X \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$

$$= \frac{1}{2}\lim_{X\to\infty}\int_0^X \frac{(x^2+a^2)+(x^2+b^2)-(a^2+b^2)}{(x^2+a^2)(x^2+b^2)}\,dx$$

$$= \frac{1}{2}\lim_{X\to\infty}\left[\int_0^X \frac{dx}{x^2+b^2} + \int_0^X \frac{dx}{x^2+a^2} - (a^2+b^2)\int_0^X \frac{dx}{(x^2+a^2)(x^2+b^2)}\right.$$

Now $\displaystyle\lim_{X\to\infty}\int_0^X \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\lim_{X\to\infty}\int_0^X\left(\frac{1}{x^2+a^2}-\frac{1}{x^2+b^2}\right)dx$

$$= \frac{1}{b^2-a^2}\lim_{X\to\infty}\left[\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{1}{b}\tan^{-1}\frac{x}{b}\right]_0^X$$

$$= \frac{1}{b^2-a^2}\lim_{X\to\infty}\left[\frac{1}{a}\tan^{-1}\frac{X}{a} - \frac{1}{b}\tan^{-1}\frac{X}{b}\right]$$

$$= \frac{1}{b^2-a^2}\left[\frac{\pi}{2a}-\frac{\pi}{2b}\right] = \frac{\pi}{2(b^2-a^2)ab}(b-a)$$

$$= \frac{\pi}{2ab(a+b)}$$

So, $\displaystyle\frac{1}{2}\lim_{X\to\infty}\left[\int_0^X \frac{dx}{x^2+b^2} + \int_0^X \frac{dx}{x^2+a^2} - (a^2+b^2)\int_0^X \frac{dx}{(x^2+a^2)(x^2+b^2)}\right]$

$$= \frac{1}{2}\lim_{X\to\infty}\left[\frac{1}{b}\tan^{-1}\frac{x}{b}\right]_0^X + \frac{1}{2}\lim_{X\to\infty}\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^X - \frac{(a^2+b^2)}{2}\,\frac{\pi}{2ab(a+b)}$$

$$= \frac{1}{2}\times\frac{\pi}{2b} + \frac{1}{2}\times\frac{\pi}{2a} - \frac{(a^2+b^2)\pi}{4ab(a+b)}$$

$$= \frac{\pi}{4ab(a+b)}\left\{(a+b)^2-(a^2+b^2)\right\} = \frac{\pi}{4ab(a+b)}\,2ab = \frac{\pi}{2(a+b)}$$