Let $x = 1-y$ Then $dx = -dy$

x	ε	$1-\delta$
y	$1-\varepsilon$	δ

So, $\lim\limits_{\substack{\varepsilon\to 0+\\ \delta\to 0+}} \int_{\varepsilon}^{1-\delta} x^{m-1}(1-x)^{n-1}\,dx = \lim\limits_{\substack{\varepsilon\to 0+\\ \delta\to 0+}} -\int_{1-\varepsilon}^{\delta} (1-y)^{m-1} y^{n-1}\,dy$

$= \lim\limits_{\substack{\varepsilon\to 0+\\ \delta\to 0+}} \int_{\delta}^{1-\varepsilon} y^{n-1}(1-y)^{m-1}\,dy$ $\left[\text{As } -\int_b^a f(x)\,dx = \int_a^b f(x)\,dx\right]$

$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = B(n, m)$

So, $B(m, n) = B(n, m)$

3. $B(m+1, n) = \dfrac{m}{m+n} B(m, n)$, $m>0$, $n>0$

Proof: $B(m+1, n) = \int_0^1 x^m (1-x)^{n-1}\,dx$

$= \left[-\dfrac{x^m (1-x)^n}{n}\right]_0^1 + \dfrac{m}{n}\int_0^1 x^{m-1}(1-x)^n\,dx$

$= \dfrac{m}{n}\int_0^1 (1-x)\,x^{m-1}(1-x)^{n-1}\,dx$

$= \dfrac{m}{n}\int_0^1 x^{m-1}(1-x)^{n-1}\,dx - \dfrac{m}{n}\int_0^1 x^m (1-x)^{n-1}\,dx$

$= \dfrac{m}{n} B(m, n) - \dfrac{m}{n} B(m+1, n)$

or, $\dfrac{m+n}{n} B(m+1, n) = \dfrac{m}{n} B(m, n)$

or, $B(m+1, n) = \dfrac{m}{m+n} B(m, n)$

4. $B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$, $m>0, n>0$

Let $x = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta\, d\theta$

As $x \to 0+$, $\theta \to 0+$ and as $x \to 1-$, $\theta \to \frac{\pi}{2}-$

So, $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \sin\theta\cos\theta\, d\theta$

$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$, $m>0, n>0$

Deduction from 4. :

(i) $\int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{1}{2}B(p,q)$ where $m = 2p-1$, $n = 2q-1$

or, $p = \frac{m+1}{2}$, $q = \frac{n+1}{2}$. So $p>0, q>0$ So, $m>-1$ and $n>-1$

So, $\int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{1}{2}B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$, $m>-1$, $n>-1$.

(ii) $\int_0^{\pi/2} \sin^n\theta = \int_0^{\pi/2}\cos^n\theta = \frac{1}{2}B\left(\frac{n+1}{2}, \frac{1}{2}\right)$, $n>-1$

(iii) $B\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = \pi$.

5. $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$, $m>0$, $n>0$

Proof: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m>0$, $n>0$

Let $x = \frac{t}{1+t}$ $dx = \frac{1}{(1+t)^2}dt$ and $t = \frac{x}{1-x}$

As $x\to 0+$, $t\to 0+$ and as $x\to 1-$, $t\to\infty$.

So, $B(m,n) = \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1}\left(1-\frac{t}{1+t}\right)^{n-1}\cdot\frac{1}{(1+t)^2}dt$

$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$ $\left[\text{As} \int_a^b f(t)\,dt = \int_a^b f(x)\,dx\right]$

6. $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

Proof: We have from 5., $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

$= \int_0^1 \frac{x^{m-1}dx}{(1+x)^{m+n}} + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

Let $x = \frac{1}{t}$ in the second integral. Then $dx = -\frac{1}{t^2}dt$

As $x \to 1+$, $t \to 1-$ and as $x \to \infty$, $t \to 0+$

So, $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = -\int_1^0 \frac{1}{t^{m-1}} \frac{t^{m+n}}{(1+t)^{m+n}}dt = \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}dt$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

So, $B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$$

The improper integral $\int_0^\infty e^{-x}x^{n-1}dx$ is convergent if $n>0$. The integral $\int_0^\infty e^{-x}x^{n-1}dx$, $n>0$ is called the Gamma Function and it is denoted by $\Gamma(n)$ or $\boxed{n}$

7. $\Gamma(1) = 1$

Proof: $\Gamma(1) = \int_0^\infty e^{-x}dx = \lim_{X\to\infty}\int_0^X e^{-x}dx = \lim_{X\to\infty}\left[-e^{-x}\right]_0^X$

$$= \lim_{X\to\infty}\left(1-e^{-X}\right) = 1$$

8. $\Gamma(n+1) = n\Gamma(n)$, $n>0$

Proof $\int_\varepsilon^X x^n e^{-x}dx = \left[-x^n e^{-x}\right]_\varepsilon^X + n\int_\varepsilon^X x^{n-1}e^{-x}dx$

$$= -X^n e^{-X} + \varepsilon^n e^{-\varepsilon} + n\int_\varepsilon^X x^{n-1}e^{-x}dx$$

Proceeding to the limit as $X\to\infty$ and $\varepsilon\to 0+$, we have

$$\int_0^\infty e^{-x}x^n dx = n\int_0^\infty e^{-x}x^{n-1} \quad \text{or} \quad \Gamma(n+1) = n\Gamma(n)$$

9. $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m>0,\ n>0$

We assume this result without proof

Deduction from 9.

(i) $\Gamma(1/2) = \sqrt{\pi}$

Proof: $B\left(\frac{1}{2},\frac{1}{2}\right) = \dfrac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)}$. So, $\left(\Gamma(1/2)\right)^2 = B\left(\frac{1}{2},\frac{1}{2}\right) = \pi$ (from 7 of 4.(iii))

So, $\Gamma(1/2) = \sqrt{\pi}$

(ii) [From result 8, if n be a positive integer

Then $\Gamma(n+1) = n!$

Proof: $\Gamma(n+1) = n\Gamma n = n(n-1)\Gamma(n-1) = \dots = n(n-1)\cdots 2\cdot 1\,\Gamma(1) = n!$]

If m, n be positive integers,

$$B(m+1, n+1) = \frac{m!\,n!}{(m+n+1)!}$$

Proof: $B(m+1, n+1) = \dfrac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} = \dfrac{m!\,n!}{(m+n+1)!}$ when m and n are positive integers.

10. (Legendre's Duplication Formula)

$$\sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\cdot\Gamma(n)\Gamma(n+1/2), \quad n>0$$

Proof: $\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n) = 2\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \quad \dots (i)$

Taking $m=n$, we have $\dfrac{(\Gamma(n))^2}{\Gamma(2n)} = 2\displaystyle\int_0^{\pi/2} \sin^{2n-1}\theta\cos^{2n-1}\theta\, d\theta$

$$= \frac{1}{2^{2n-2}}\int_0^{\pi/2} \sin^{2n-1} 2\theta\, d\theta = \frac{1}{2^{2n-1}}\int_0^{\pi} \sin^{2n-1}\phi\, d\phi \quad [\text{Taking } 2\theta = \phi]$$

$$= \frac{1}{2^{2n-1}}\cdot 2\int_0^{\pi/2} \sin^{2n-1}\theta\, d\theta \quad \left[\text{As } \int_a^b f(\phi)\,d\phi = \int_a^b f(\theta)\,d\theta\right] \quad \dots (ii)$$

From (i), taking $m = \frac{1}{2}$, we have $\dfrac{\Gamma(n)\Gamma(1/2)}{\Gamma(n+1/2)} = 2\displaystyle\int_0^{\pi/2} \cos^{2n-1}\theta\, d\theta$

$= 2\displaystyle\int_0^{\pi/2} \sin^{2n-1}\theta\, d\theta \quad \dots (iii)$. So, from (ii) & (iii), we have

$\dfrac{(\Gamma(n))^2}{\Gamma(2n)} = \dfrac{1}{2^{2n-1}}\dfrac{\Gamma(n)\Gamma(1/2)}{\Gamma(n+1/2)}$. Or, $\sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+1/2)$, Since $\Gamma(1/2) = \sqrt{\pi}$