11. $\Gamma(m)\,\Gamma(1-m) = \dfrac{\pi}{\sin m\pi} = \pi \operatorname{cosec} m\pi, \quad 0 < m < 1$

We assume the above result also without proof.

Worked out Exercises

1. Show that $B(m, 1) = \dfrac{1}{m}$, $m > 0$

Solution: $B(m,1) = \displaystyle\int_0^1 x^{m-1}\,dx$, $m > 0$

$$= \lim_{\varepsilon \to 0+} \int_\varepsilon^1 x^{m-1}\,dx = \lim_{\varepsilon \to 0+} \left[\frac{x^m}{m}\right]_\varepsilon^1 = \frac{1}{m} \lim_{\varepsilon \to 0+} \left(1 - \varepsilon^m\right) = \frac{1}{m}$$

2. Show that

(i) $\Gamma(6) = 120$

(ii) $\Gamma(5/2) = \dfrac{3}{4}\sqrt{\pi}$

(iii) $\Gamma(9/2) = \dfrac{105}{16}\sqrt{\pi}$

Solution: (i) $\Gamma(6) = 5! = 120$ (as $\Gamma n = (n-1)!$, if n be a positive integer)

(ii) $\Gamma(5/2) = \dfrac{3}{2}\Gamma(3/2) = \dfrac{3}{2}\cdot\dfrac{1}{2}\Gamma(1/2) = \dfrac{3}{4}\sqrt{\pi}$ (as $\Gamma(1/2) = \sqrt{\pi}$)

(iii) $\Gamma(9/2) = \dfrac{7}{2}\Gamma(7/2) = \dfrac{7}{2}\cdot\dfrac{5}{2}\cdot\Gamma(5/2) = \dfrac{7}{2}\cdot\dfrac{5}{2}\cdot\dfrac{3}{2}\Gamma(3/2) = \dfrac{7}{2}\cdot\dfrac{5}{2}\cdot\dfrac{3}{2}\cdot\dfrac{1}{2}\Gamma(1/2) = \dfrac{105}{16}\sqrt{\pi}$ (as $\Gamma(1/2) = \sqrt{\pi}$)

3. Show that $\dfrac{B(p,q)}{p+q} = \dfrac{B(p+1,q)}{p} = \dfrac{B(p,q+1)}{q}$

Solution: $\dfrac{B(p,q)}{p+q} = \dfrac{\Gamma(p)\,\Gamma(q)}{(p+q)\,\Gamma(p+q)}$.

$$\frac{B(p+1,q)}{p} = \frac{\Gamma(p+1)\,\Gamma(q)}{p\,\Gamma(p+q+1)} = \frac{p\,\Gamma(p)\,\Gamma(q)}{p\,(p+q)\,\Gamma(p+q)} = \frac{\Gamma(p)\,\Gamma(q)}{(p+q)\,\Gamma(p+q)}$$

and $$\frac{B(p,q+1)}{q} = \frac{\Gamma(p)\,\Gamma(q+1)}{q\,\Gamma(p+q+1)} = \frac{\Gamma(p)\cdot q\,\Gamma(q)}{q\,(p+q)\,\Gamma(p+q)} = \frac{\Gamma(p)\,\Gamma(q)}{(p+q)\,\Gamma(p+q)}$$

So, $$\frac{B(p,q)}{p+q} = \frac{B(p+1,q)}{p} = \frac{B(p,q+1)}{q}$$

4. Prove that (i) $\int_0^\infty e^{-kt} t^{n-1} dt = \frac{\Gamma(n)}{k^n}$, $k>0, n>0$

(ii) $\int_1^\infty \frac{(\log y)^{n-1}}{y^{k+1}} dy = \frac{\Gamma(n)}{k^n}$, $k>0, n>0$

Proof: (i) $\int_0^\infty e^{-kt} t^{n-1} dt$

$= \frac{1}{k}\int_0^\infty e^{-y} \left(\frac{y}{k}\right)^{n-1} dy$ [Let $y = kt$. As $t \to \infty$, $y \to \infty$ since $k>0$]

$= \frac{1}{k^n} \int_0^\infty e^{-y} y^{n-1} dy$

$= \frac{\Gamma(n)}{k^n}$

(ii) $\int_1^\infty \frac{(\log y)^{n-1}}{y^{k+1}} dy$

$= \int_0^\infty \frac{t^{n-1} e^t dt}{(e^t)^{k+1}}$ [Let $\log y = t$. Then $y = e^t$. $y=1 \Rightarrow t=0$]

$= \int_0^\infty t^{n-1} e^{-kt} dt$

$= \frac{\Gamma(n)}{k^n}$ (from (i))

5. Prove that $\int_0^\infty e^{-x^2} dx$

Solution: Let $x^2 = t$. Then $2x\,dx = dt$ or, $dx = \frac{1}{2\sqrt{t}} dt$

As $x \to \infty$, $t \to \infty$

So, $\int_0^\infty e^{-x^2} dx = \frac{1}{2} \int_0^\infty e^{-t} t^{-1/2} dt = \frac{1}{2}\Gamma\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$

6. Show that $\int_0^\infty 5^{-x^2} dx = \frac{1}{2\sqrt{\log 5}}\sqrt{\pi}$

Solution: $\int_0^\infty 5^{-x^2} dx = \int_0^\infty e^{-x^2 \log 5} dx$

Let $x^2 \log 5 = u$. So, $2\log 5 . x\,dx = du$

$u = 0$ when $x = 0$ and $u \to \infty$ when $x \to \infty$

So, $$\int_0^\infty 5^{-x^2} dx = \int_0^\infty e^{-u} \cdot \frac{du}{2\log 5 \sqrt{\frac{u}{\log 5}}}$$

$$= \frac{1}{2\sqrt{\log 5}} \int_0^\infty e^{-u} u^{\frac{1}{2}-1} du = \frac{1}{2\sqrt{\log 5}} \Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2\sqrt{\log 5}}$$

7. Prove that

$$\Gamma(\tfrac{1}{9})\Gamma(\tfrac{2}{9}) \cdots \Gamma(\tfrac{8}{9}) = \frac{16}{3}\pi^4$$

Proof: $\Gamma(\tfrac{1}{9})\Gamma(\tfrac{2}{9}) \cdots \Gamma(\tfrac{8}{9})$

$$= \Gamma(\tfrac{1}{9})\Gamma(\tfrac{8}{9})\Gamma(\tfrac{2}{9})\Gamma(\tfrac{7}{9})\Gamma(\tfrac{3}{9})\Gamma(\tfrac{6}{9})\Gamma(\tfrac{4}{9})\Gamma(\tfrac{5}{9})$$

$$= \Gamma(\tfrac{1}{9})\Gamma(1-\tfrac{1}{9})\Gamma(\tfrac{2}{9})\Gamma(1-\tfrac{2}{9})\Gamma(\tfrac{3}{9})\Gamma(1-\tfrac{3}{9})\Gamma(\tfrac{4}{9})\Gamma(1-\tfrac{4}{9})$$

$$= \pi \operatorname{cosec}\tfrac{\pi}{9} \cdot \pi \operatorname{cosec}\tfrac{2\pi}{9} \cdot \pi \operatorname{cosec}\tfrac{3\pi}{9} \cdot \pi \operatorname{cosec}\tfrac{4\pi}{9} \quad \left[\text{As } \Gamma(m)\Gamma(1-m) = \pi \operatorname{cosec} m\pi,\ 0 < m < 1\right]$$

$$= \frac{2\pi^4}{\sqrt{3}} \operatorname{cosec}\frac{\pi}{9}\operatorname{cosec}\frac{2\pi}{9}\operatorname{cosec}\frac{4\pi}{9} \quad \left[\because \operatorname{cosec}\frac{3\pi}{9} = \operatorname{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}}\right]$$

$$= \frac{2\pi^4}{\sqrt{3}} \frac{1}{\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{4\pi}{9}} = \frac{2\pi^4}{\sqrt{3}} \frac{2}{\left(\cos\frac{\pi}{9} - \cos\frac{\pi}{3}\right)\sin\frac{4\pi}{9}}$$

$$= \frac{4\pi^4}{\sqrt{3}} \cdot \frac{2}{2\cos\frac{\pi}{9}\sin\frac{4\pi}{9} - \sin\frac{4\pi}{9}} = \frac{8\pi^4}{\sqrt{3}} \frac{1}{\sin\frac{5\pi}{9} + \sin\frac{3\pi}{9} - \sin\frac{4\pi}{9}}$$

$$= \frac{8\pi^4}{\sqrt{3}} \frac{1}{\frac{\sqrt{3}}{2} + \sin\frac{5\pi}{9} - \sin\frac{4\pi}{9}} = \frac{8\pi^4}{\sqrt{3}} \cdot \frac{1}{\frac{\sqrt{3}}{2} + 2\cos\pi/2 \sin\frac{\pi}{18}} = \frac{8\pi^4}{\sqrt{3} \cdot \frac{\sqrt{3}}{2}}$$

$$= \frac{16\pi^4}{3}$$

8. Show that $$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4} \frac{\Gamma(\tfrac{1}{4}) \cdot \sqrt{\pi}}{\Gamma(\tfrac{3}{4})}$$

Solution: Let $x^2 = \sin\theta$. So, $2x\,dx = \cos\theta\,d\theta$

x	0	1
θ	0	$\pi/2$

So, $$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{2}\int_0^{\pi/2} \frac{\cos\theta\,d\theta}{\sqrt{\sin\theta}\,\cos\theta} = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\,d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}B\left(\frac{1}{4},\frac{1}{2}\right) \quad \left[\because \int_0^{\pi/2}\sin^m\theta\cos^n\theta\,d\theta = \frac{1}{2}B\left(\frac{m+1}{2},\frac{n+1}{2}\right),\ m>-1,\ n>-1\right]$$

$$= \frac{1}{2}\,\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)} = \frac{1}{2}\,\frac{\Gamma(1/4)\sqrt{\pi}}{\Gamma(3/4)} \quad \left[\because \Gamma\tfrac{1}{2} = \sqrt{\pi}\right]$$

9. Show that $\displaystyle\int_0^1 \frac{dx}{(1-x^6)^{1/6}} = \pi/3$

Solution: Let $x^3 = \sin\theta$ $\therefore 3x^2\,dx = \cos\theta\,d\theta$

x	0	1
θ	0	$\pi/2$

So, $$\int_0^1 \frac{dx}{(1-x^6)^{1/6}} = \frac{1}{3}\int_0^{\pi/2} \frac{\cos\theta\,d\theta}{\sin^{2/3}\theta\,\cos^{1/3}\theta}$$

$$= \frac{1}{3}\int_0^{\pi/2} \sin^{-2/3}\theta\cdot\cos^{2/3}\theta\,d\theta$$

$$= \frac{1}{3}\cdot\frac{1}{2}B\left(\frac{-2/3+1}{2},\frac{2/3+1}{2}\right) = \frac{1}{6}B\left(\frac{1}{6},\frac{5}{6}\right)$$

$$= \frac{1}{6}\,\frac{\Gamma(1/6)\,\Gamma(5/6)}{\Gamma(1)} = \frac{1}{6}\Gamma\tfrac{1}{6}\,\Gamma\left(1-\tfrac{1}{6}\right) = \frac{1}{6}\pi\cdot\mathrm{cosec}\frac{\pi}{6} = \frac{1}{6}\cdot\pi\cdot 2 = \frac{\pi}{3}$$

$\left[\because \Gamma(1) = 1 \text{ and } \Gamma(m)\Gamma(1-m) = \pi\,\mathrm{cosec}\,m\pi,\ 0<m<1\right]$

Exercises 1. Show that

(i) $B(m,n)\cdot B(m+n,l) = B(n,l)\cdot B(n+l,m) = B(l,m)\cdot B(l+m,n)$

(ii) $\displaystyle\int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi}$

(iii) $\displaystyle\int_0^{\pi/2} \sin^4 x\cos^4 x\,dx = \frac{3\pi}{256}$

2. Show that $\displaystyle\int_0^1 x^{m-1}(1-x^2)^{n-1}\,dx = \frac{1}{2}B\left(\frac{1}{2}m, n\right)$; $m, n > 0$

3. Show that $\displaystyle\int_0^\infty e^{-x^2}x^m\,dx = \frac{1}{2}\Gamma\left(\frac{m+1}{2}\right)$; $m > -1$