4. Show that $\int_0^{\pi/2} \sqrt{\tan x}\, dx = \frac{\pi}{\sqrt{2}}$

5. Show that $\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \cdot \int_0^{\pi/2} \sqrt{\sin x}\, dx = \pi$

6. Show that $\int_0^1 \frac{x^2 dx}{(1-x^4)^{1/2}} \cdot \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$

7. Show that $\int_0^1 x^{m-1}(1-x)^{n-1} dx = \frac{(m-1)!}{n(n+1)\cdots(n+m-1)}$,

where $m$ is a positive integer and $n > 0$

8. Substituting $x = a\cos^2\theta + b\sin^2\theta$, show that

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1} dx = (b-a)^{m+n-1} \cdot B(m,n)\ ; \quad m, n > 0$$

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## Double Integration

The concept of double integral is an extension of the concept of to the case of two variables.

Let $E$ be a bounded region in $\mathbb{R}^2$ (i.e., in two dimensional plane) and $f: E \to \mathbb{R}$ be a ~~d~~ bounded function on $E$ and $A$ be the area of $E$

We divide the region $E$ into $n$ subregions of ~~sa~~ areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$. Let $(x_i, y_i)$ be any point on the $i$th subregion of area $\Delta A_i$, $i = 1, 2, \ldots, n$.

$$\text{Let } S = \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i \quad .$$

Let $\Delta A_r$ = the largest area of $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$

If $\lim\limits_{\Delta A_r \to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$ exists for ~~any~~ all $(x_i, y_i)$ in the subregion of area $\Delta A_i$ and for all division of $E$ into subregions, then we say $\iint_E f(x,y)\,dx\,dy$ exists and $\iint_E f(x,y)\,dx\,dy = \lim_{\Delta A_r \to 0} S$

Properties of a double integral

1. If the region E is partitioned into two parts, say $E_1$ and $E_2$,

Then $\iint_E f(x,y)\,dx\,dy = \iint_{E_1} f(x,y)\,dx\,dy + \iint_{E_2} f(x,y)\,dx\,dy$

Similar is the case for a subdivision of E into three or parts.

2. ~~The algebraic sum~~ The double integral of the algebraic sum of a fixed number of functions is equal to the algebraic sum of the double integrals taken for each term. So,

$$\iint_E \left(f_1(x,y) + f_2(x,y) + f_3(x,y) + \dots + f_n(x,y)\right) dx\,dy$$

$$= \iint_E f_1(x,y)\,dx\,dy + \iint_E f_2(x,y)\,dx\,dy + \iint_E f_3(x,y)\,dx\,dy + \dots + \iint_E f_n(x,y)\,dx\,dy$$

3. A constant factor may be taken outside the integral sign. So,

$$\iint_E m f(x,y)\,dx\,dy = m \iint_E f(x,y)\,dx\,dy, \quad m \text{ is a constant.}$$

Evaluation of Double Integrals

(a) If the region E be given by $\{(x,y) : a \le x \le b,\ c \le y \le d\}$

Then $\iint_E f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dx\,dy$

$$= \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx \quad \dots (1)$$

or, $\iint_E f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dy\,dx$

$$= \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy \quad \dots (2)$$

i.e., in this case the order of ~~the~~ integration is ~~immater~~ immaterial provided the limits of integrations are changed accordingly.

Note: In formula (1), the definite integral $\int_c^d f(x,y)\,dy$ is calculated first. During this integration, $x$ is regarded as a constant. While in the formula (2), the definite integral $\int_a^b f(x,y)\,dx$ is calculated first and during the integration $y$ is regarded as a constant.

(b) If the region $E$ is bounded by the curves $y=f_1(x)$, $y=f_2(x)$, $x=a$, $x=b$, then

$$\iint_E f(x,y)\,dx\,dy = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\right] dx$$

where the integration with respect to $y$ is ~~perform~~ performed first treating $x$ as a constant.

Similarly, if the region $E$ is bounded by the curves $x=f_1(y)$, $x=f_2(y)$, $y=c$, $y=d$, we have

$$\iint_E f(x,y)\,dx\,dy = \int_c^d \int_{f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{f_1(y)}^{f_2(y)} f(x,y)\,dx\right] dy$$

where the integration with respect to $x$ is performed first treating $y$ as a constant.

Remember: While evaluating double integrals, first integrate with respect to the variable having variable limits (treating the other variable as constant) and then integrate with respect to the variable with constant limits

Change of order of Integration: If in a double integral, the limits of integration of both $x$ any $y$ are constant, we can generally integrate $\iint f(x,y)\,dx\,dy$ in either order. But if the limits of $y$ are functions of $x$, we must first integrate with respect to $y$ ~~regard~~ regarding $x$ as constant and then integrate with respect to $x$. In this case the order of integration can be changed only if we find the new limits of $x$ as functions of $y$ and the new constant limit of $y$.

## Change of variables in a double integral

Sometimes, the evaluation of a double integral becomes more convenient by a suitable change of variables from one system to another.

Let the variables in the double integral $\iint_E f(x,y)\,dx\,dy$ be changed from $x, y$ to $u, v$ where

$$x = \phi(u,v) \quad y = \psi(u,v)$$

Then on substituting for $x$ and $y$, the double integral is transformed to $\iint_{E'} F(u,v)\, J\, du\, dv$

where $J$ is the Jacobian of $x, y$ with respect to $u, v$,

i.e., $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$ where $E'$ is the region in the uv-plane corresponding to the region $E$ in the xy-plane.

Special Case: Change to polar co-ordinates from Cartesian co-ordinates, we put $x = r\cos\theta$, $y = r\sin\theta$. In this case

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

and therefore $dx\,dy = J\,dr\,d\theta$. This change is specially useful when the region of integration is a circle or part of a circle.