Remark: In the double integral $\int_a^b \int_c^d f(x,y)\,dx\,dy$, it is generally understood that that limits of integration c to d are those of y and the limits of integration a to b are those of x and here also we will understand the same thing. However some authors regard these limits in the reverse order, i.e., they regard the limits c to d as those of x and a to b as those of y. So, it is better to write this double integral as $\int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dx\,dy$ so that there is no confusion about the limits. However in the double integral $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy$, there is no confusion about the limits. Obviously, the variable limits are those of y because they are in terms of x and so that the constant limits must be those of x. Here the first integration must be performed with respect to y regarding x as constant.

Worked out Exercises:

1. Show that

(i) $\int_0^4 \int_0^1 xy(x-y)\,dy\,dx = 8$

(ii) $\int_0^{\pi} \int_0^{a(1-\cos\theta)} r\,dr\,d\theta = \frac{3}{4}\pi a^2$

(iii) $\int_0^1 dx \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy = \frac{\pi}{6}$

(iv) $\int_0^1 dx \int_0^x \frac{y\,dy}{\sqrt{x^2+y^2}} = \frac{\sqrt{2}-1}{2}$

Solution: (i) $\int_0^4 \int_0^1 xy(x-y)\,dy\,dx = \int_0^4 x\left[\frac{xy^2}{2} - \frac{y^3}{3}\right]_0^1 dx$

$$= \int_0^4 x\left(\frac{x}{2}-\frac{1}{3}\right)dx = \left[\frac{x^3}{6}-\frac{x^2}{6}\right]_0^4 = \frac{1}{6}\left[64-16\right] = \frac{1}{6}\cdot 48 = 8.$$

(ii) $$\int_0^{\pi}\int_0^{a(1-\cos\theta)} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi}\left[r^2\right]_0^{a(1-\cos\theta)} d\theta = \frac{1}{2}\int_0^{\pi} a^2(1-\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta = \frac{a^2\pi}{2}+\frac{a^2}{4}\int_0^{\pi}(1+\cos 2\theta)\,d\theta$$

$$= \frac{a^2\pi}{2}+\frac{\pi a^2}{4} = \frac{3\pi a^2}{4} = \frac{3}{4}\pi a^2$$

(iii) $$\int_0^1 dx \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy$$

$$= \int_0^1 \left[\frac{y}{2}\sqrt{1-x^2-y^2} + \frac{1-x^2}{2}\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1-x^2}{2}\cdot\frac{\pi}{2}\,dx = \frac{\pi}{4}\left[x-\frac{x^3}{3}\right]_0^1 = \frac{\pi}{4}\cdot\frac{2}{3} = \frac{\pi}{6}$$

(iv) $$\int_0^1 dx\int_0^x \frac{y\,dy}{\sqrt{x^2+y^2}} = \int_0^1 \left[\sqrt{x^2+y^2}\right]_0^x dx$$

$$= \int_0^1 (x\sqrt{2}-x)\,dx = (\sqrt{2}-1)\left[\frac{x^2}{2}\right]_0^1 = \frac{\sqrt{2}-1}{2}$$

Exercise 2. Show that

(i) $$\int_0^1\int_0^{1-y^2}\{(x-1)^2+y^2\}\,dx\,dy = \frac{44}{105}$$

(ii) $$\cancel{\int_0^{\pi/2}\int_0^{2(1+\cos\theta)} r\,dr\,d\theta} = \cancel{\frac{3\pi}{2}} \int_0^{2\pi}\int_{a\sin\theta}^{a} r\,dr\,d\theta = \frac{\pi a^2}{2}$$

(iii) $$\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dy\,dx = \frac{1}{8}$$

Worked out exercises 3. Show that

(i) $\iint_E \sin(x+y)\,dx\,dy$ over E, where E is $\{0\le x\le \pi/2,\ 0\le y\le \pi/2\}$, is 2

(ii) ~~[illegible] over [illegible]~~ is $\int_0^{\pi/2}\int_0^{2(1+\cos\theta)} r\,dr\,d\theta = \frac{3\pi}{2}+4$

(iii) $\int_0^{\log 2} dx \int_{-1}^{1} y e^{xy}\,dy = \frac{3}{2\log 2} - 2$

Solution : (i) $\iint_E \sin(x+y)\,dx\,dy = \int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\,dy\,dx$

$$= -\int_0^{\pi/2}\Big[\cos(x+y)\Big]_0^{\pi/2} dx = -\int_0^{\pi/2}\Big[\cos(\tfrac{\pi}{2}+x) - \cos x\Big]dx$$

$$= \int_0^{\pi/2}(\sin x + \cos x)\,dx = \Big[\sin x - \cos x\Big]_0^{\pi/2} = 2$$

(ii) $\int_0^{\pi/2}\int_0^{2(1+\cos\theta)} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\Big[r^2\Big]_0^{2(1+\cos\theta)} d\theta$

$$= \frac{1}{2}\int_0^{\pi/2} 4\,(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= 2\left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi/2}$$

$$= 2\left[\frac{\pi}{2} + 2 + \frac{\pi}{4}\right] = \frac{3\pi}{2} + 4$$

(iii) $\int_0^{\log 2} dx \int_{-1}^{1} y e^{xy}\,dy = \int_{-1}^{1} y\left\{\int_0^{\log 2} e^{xy}\,dx\right\}dy = \int_{-1}^{1} y\left[\frac{e^{xy}}{y}\right]_0^{\log 2} dy$

$$= \int_{-1}^{1}\left(e^{y\log 2} - 1\right)dy = \left[\frac{e^{y\log 2}}{\log 2} - y\right]_{-1}^{1} = \frac{2}{\log 2} - 1 - \left(\frac{1}{2\log 2} + 1\right)$$

$$= \frac{3}{2\log 2} - 2$$

Exercise 4. Show that

(i) $\iint_R xy(x^2+y^2)\,dx\,dy$ where $R: \{0 \le x \le a,\ 0 \le y \le b\}$ is $\frac{1}{8}a^2b^2(a^2+b^2)$

(ii) $\iint_R x^3 y\,dx\,dy$ over $R: \{0 \le x \le 1,\ 0 \le y \le 2\}$ is $\frac{1}{2}$

Worked out exercise #5. Evaluate the following integrals over R :

(i) $\iint_R \frac{x}{y}\,dx\,dy$, $R: \{|x| \le 1;\ 1 \le y \le 2\}$

(ii) $\iint_R x\sin(x+y)\,dx\,dy$, $R: \{0 \le x \le \pi,\ 0 \le y \le \pi/2\}$

(iii) ~~$\iint_R \frac{dx\,dy}{\sqrt{x^2+y^2}}$, $R: \{|x| \le 1, |y| \le 1\}$~~ $\iint_R x e^{xy}\,dx\,dy$, $R = \{(x,y): 0 \le x \le b, 0 \le y \le a\}$

(iv) $\iint_R (x^2+y^2)\,dx\,dy$, $R: \{(x,y): 0 \le x \le 1,\ 1 \le y \le 2\}$

Solution: (i) $\iint_R \frac{x}{y}\,dx\,dy = \int_1^2 \int_{-1}^1 \frac{x}{y}\,dx\,dy = \int_1^2 \left[\frac{x^2}{2y}\right]_{-1}^1 dy$

$= \int_1^2 0\,dy = 0$

(ii) $\iint_R x\sin(x+y)\,dx\,dy = \int_0^\pi \int_0^{\pi/2} x\sin(x+y)\,dy\,dx$

$= -\int_0^\pi x\left[\cos(x+y)\right]_0^{\pi/2} dx = -\int_0^\pi x(-\sin x - \cos x)\,dx$

$= \int_0^\pi x(\sin x + \cos x)\,dx$

$= \left[x(\sin x - \cos x)\right]_0^\pi - \int_0^\pi (\sin x - \cos x)\,dx$

$= \pi + \left[\cos x + \sin x\right]_0^\pi = \pi - 2$

(iii) $\iint_R x e^{xy}\,dx\,dy = \int_0^b x\left\{\int_0^a e^{xy}\,dy\right\} dx = \int_0^b x\left[\frac{e^{xy}}{x}\right]_0^a dx$

$= \int_0^b (e^{ax}-1)\,dx = \left[\frac{e^{ax}}{a} - x\right]_0^b = \frac{1}{a}\left(e^{ab} - ab - 1\right)$