(iv) $\iint_R (x^2+y^2)\,dx\,dy = \int_0^1\int_1^2 (x^2+y^2)\,dx\,dy$

$$= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_1^2 dx = \int_0^1 \left(x^2 + \frac{7}{3}\right)dx$$

$$= \left[\frac{x^3}{3} + \frac{7}{3}x\right]_0^1 = \frac{1}{3} + \frac{7}{3} = \frac{8}{3}$$

Exercise 6. Evaluate the following integrals over R :

(i) $\iint_R \frac{dx\,dy}{\sqrt{x^2+y^2}}$, $R : \{|x| \le 1, |y| \le 1\}$

(ii) $\iint_R \frac{dx\,dy}{(x+y)^2}$, $R : \{3 \le x \le 4, 1 \le y \le 2\}$

(iii) $\iint_R (x+y)\,dx\,dy$, $R : \{1 \le x \le 2, 0 \le y \le 1\}$

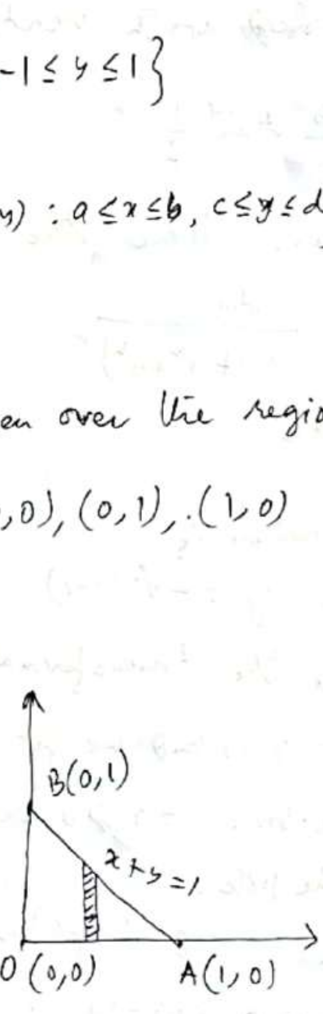
(iv) $\iint_R (x+y)^2\,dx\,dy$, $R = \{(x,y) : -1 \le x \le 1, -1 \le y \le 1\}$

(note : $R : \{a \le x \le b, c \le y \le d\}$ implies $R = \{(x,y) : a \le x \le b, c \le y \le d\}$)

Worked out Exercises :

7. Show that the integral $\iint e^{\frac{y-x}{y+x}}\,dx\,dy$ taken over the region enclosed by the triangle with vertices at $(0,0), (0,1), (1,0)$ is $\frac{1}{4}\left(e - \frac{1}{e}\right)$

Solution: Here the required integral is $\iint_{\Delta OAB} e^{\frac{y-x}{y+x}}\,dx\,dy$ where the triangular region ΔOAB is shown in figure 1.



Changing the variables x, y to u, v by the relation $x+y = u$, $x = uv$, or, $x = uv$, $y = u(1-v)$

Figure 1

Now $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & 1-v \\ u & -u \end{vmatrix} = -u$

and the region ΔOAB changes to $R' : \{0 \le u \le 1,\ 0 \le v \le 1\}$

So, $$\iint_{\Delta OAB} e^{\frac{y-x}{y+x}}\,dx\,dy = \iint_{R'} e^{\frac{u-uv-uv}{u-uv+uv}} |J|\,du\,dv$$

$$= \iint_{R'} u e^{1-2v}\,du\,dv = \int_0^1 u\,du \int_0^1 e^{1-2v}\,dv$$

$$= \frac{1}{2}\left[\frac{e^{1-2v}}{-2}\right]_0^1 = \frac{1}{4}\left[e - \frac{1}{e}\right] \quad \text{(Proved)}$$

8. Show that $\iint \frac{dx}{(1+x^2+y^2)^2}$ taken over the triangular region ~~bounded~~ with vertices at $(0,0)$, $(2,0)$, $(1,\sqrt{3})$ is $\frac{\sqrt{3}}{2}\tan^{-1}\frac{1}{2}$

Solution: Here the required integral is

$$\iint_{\Delta OAB} \frac{dx}{(1+x^2+y^2)^2}$$ where ΔOAB is given in figure 1

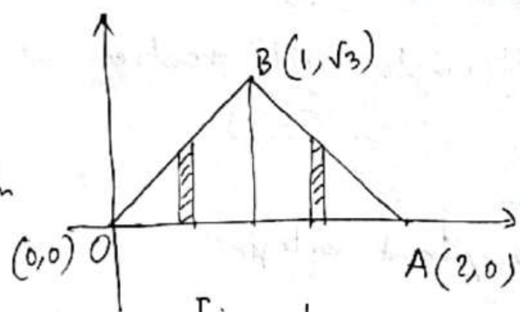


Figure 1

Here equation of AB is $y = -\sqrt{3}(x-2)$

Applying the transformation $x = r\cos\theta$, $y = r\sin\theta$ we get the Jacobian $J = r > 0$ except at the pole.

Equation of AB in polar coordinates is $r\sin\theta = -\sqrt{3}(r\cos\theta - 2)$ or, $r = \sqrt{3}\sec(\theta - \pi/6)$

The transformed region is $\Delta OA'B'$ in figure 2.

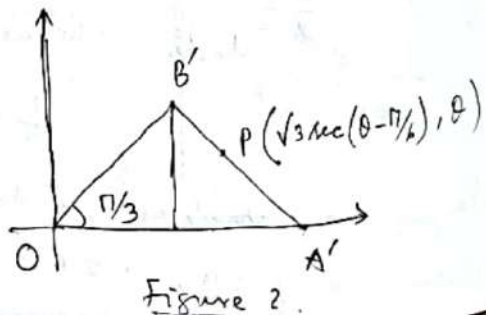


Figure 2

So, $$\iint_{\Delta OAB} \frac{dx}{(1+x^2+y^2)^2} = \iint_{\Delta OA'B'} \frac{r\,d\theta\,dr}{(1+r^2)^2}$$

$$= \int_0^{\pi/3}\left[\int_0^{\sqrt{3}\sec(\theta-\pi/6)} \frac{r\,dr}{(1+r^2)^2}\right]d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3}\left[-\frac{1}{1+r^2}\right]_0^{\sqrt{3}\sec(\theta-\pi/6)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3}\left(1-\frac{1}{1+3\sec^2(\theta-\pi/6)}\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3}\frac{3\sec^2(\theta-\pi/6)\,d\theta}{4+3\tan^2(\theta-\pi/6)}$$

Let $\tan(\theta-\pi/6) = u$

So, $\sec^2(\theta-\pi/6)\,d\theta = du$

and

θ	0	$\pi/3$
u	$-\frac{1}{\sqrt{3}}$	$1/\sqrt{3}$

$$= \frac{1}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{du}{\left(\frac{2}{\sqrt{3}}\right)^2+u^2}$$

$$= \int_0^{1/\sqrt{3}}\frac{du}{(2/\sqrt{3})^2+u^2}$$ as the integrand is an even function.

$$= \frac{\sqrt{3}}{2}\left[\tan^{-1}\frac{\sqrt{3}u}{2}\right]_0^{1/\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2}\tan^{-1}\frac{1}{2} \quad \text{(Proved)}$$

9. Show that $\iint (x^2+y^2)\,dx\,dy$ over the region bounded by $x \geq 0$, $y \geq 0$ and $x+y \leq 1$ is $\frac{1}{6}$

Solution : Here the required integral is

$\iint_{\Delta OAB} (x^2+y^2)\,dx\,dy$, ΔOAB is the region shown in figure 1.

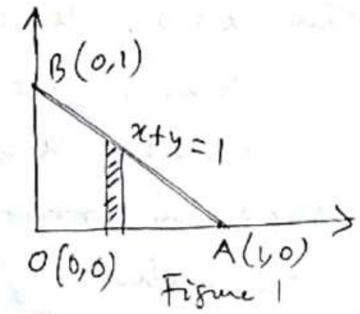


Figure 1

So, $$\iint_{\Delta OAB} (x^2+y^2)\,dx\,dy = \int_0^1 \int_0^{1-x} (x^2+y^2)\,dx\,dy = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left\{x^2(1-x) + \frac{1}{3}(1-x)^3\right\} dx$$

$$= \frac{1}{3}\int_0^1 \left(3x^2 - 3x^3 + 1 - 3x + 3x^2 - x^3\right) dx$$

$$= \frac{1}{3}\int_0^1 \left(-4x^3 + 6x^2 - 3x + 1\right) dx$$

$$= \frac{1}{3}\left[-x^4 + 2x^3 - \frac{3}{2}x^2 + x\right]_0^1$$

$$= \frac{1}{3}\left(-1 + 2 - \frac{3}{2} + 1\right) = \frac{1}{6} \quad \text{(Proved)}$$

10. Show that

$$\iint x^{m-1} y^{n-1} (1-x-y)^{l-1}\,dx\,dy = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}\ ;\quad l, m, n > 0$$

and ~~Also~~ show that $\iint \sqrt{xy(1-x-y)}\,dx\,dy = \frac{2\pi}{105}$ over the ~~region~~ triangular region bounded by $x=0,\ y=0,\ x+y=1$

Solution: Here the required integral is

$$\iint_{\Delta OAB} x^{m-1} y^{n-1} (1-x-y)^{l-1}\,dx\,dy\ ,\ \Delta OAB$$

is the region in figure 1

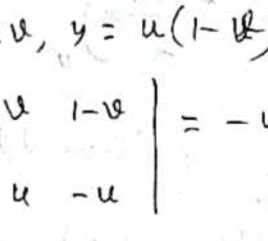


Figure 1

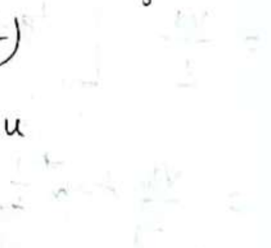


Figure 2

Change the variable x, y into u, v such that $x+y=u$, $x=uv$; or, $x=uv$, $y=u(1-v)$

Then Jacobian $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} v & 1-v \\ u & -u \end{vmatrix} = -u$

When $x=0$, $u=0$, $v=0$

$y=0$, $u=0$, $v=1$

$x+y=1$, $u=1$

So ΔOAB is transformed to the square $R_1 : \{0 \le u \le 1,\ 0 \le v \le 1\}$ as shown in Figure 2. in the uv plane