So $\iint\limits_{\Delta OAB} x^{m-1} y^{n-1} (1-x-y)^{l-1}\, dx\, dy$

$$= \iint\limits_{R_1} (uv)^{m-1} u^{n-1} (1-v)^{n-1} (1-u)^{l-1} |J|\, du\, dv$$

$$= \int_0^1 u^{m+n-1} (1-u)^{l-1} du \times \int_0^1 v^{m-1} (1-v)^{n-1} dv$$

$$= B(m+n, l) \times B(m,n)$$

$$= \frac{\Gamma(m+n)\,\Gamma(l)}{\Gamma(m+n+l)} \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \quad l, m, n > 0 \quad \text{(Proved)}$$

Now $\iint\limits_{\Delta OAB} \sqrt{xy(1-x-y)}\, dx\, dy = \iint\limits_{\Delta OAB} x^{3/2-1} y^{3/2-1} (1-x-y)^{3/2-1} dx\, dy$

$$= \frac{\Gamma(3/2)\,\Gamma(3/2)\,\Gamma(3/2)}{\Gamma(9/2)} \quad \text{from the above result.}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{\frac{7}{2}\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{2\pi}{105} \text{ (Proved)} \quad \text{as } \Gamma(l+1) = l\,\Gamma(l) \text{ and } \Gamma(1/2) = \sqrt{\pi}$$

Exercises: 11. Show that $\iint (x^2+y^2)\, dx\, dy$ over the region enclosed by the triangle having its vertices at $(0,0), (1,0), (1,1)$ is $\frac{1}{3}$

12. Show that $\iint \sqrt{4x^2-y^2}\, dx\, dy$ over the triangular region formed by $y=0, x=1, y=x$ is $(3\sqrt{3}+2\pi)/18$

13. Verify that $\iint (x^3+y^2)\, dx\, dy$ taken over the triangular region bounded by $x=2, y=0, y=2x$ is $\frac{352}{15}$

14. Show that $\iint x^{1/2} y^{1/3} (1-x-y)^{2/3}\, dx\, dy$ over the triangular region bounded by $x=0, y=0, x+y=1$ is

$B\left(\frac{17}{6}, \frac{5}{3}\right) \times B\left(\frac{4}{3}, \frac{3}{2}\right)$

Worked out Exercises

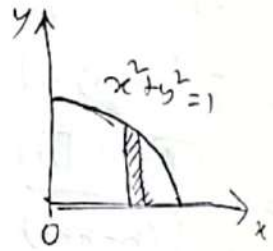
15. Evaluate $\iint x^2y^2\,dx\,dy$ over

(i) the region bounded by $x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 1$

(ii) the region within the circle $x^2+y^2=1$

Solution:

(i) The required integral



is $= \int_0^1 \int_0^{\sqrt{1-x^2}} x^2y^2\,dx\,dy = \int_0^1 x^2 \left[\frac{y^3}{3}\right]_0^{\sqrt{1-x^2}} dx$

$= \int_0^1 \frac{x^2}{3}\left(1-x^2\right)^{3/2} dx$

Let $x^2 = u$. So, $2x\,dx = du$

x	0	1
u	0	1

So $\int_0^1 \frac{x^2}{3}(1-x^2)^{3/2} dx = \frac{1}{3}\int u\,(1-u)^{3/2}\frac{du}{2\sqrt{u}}$

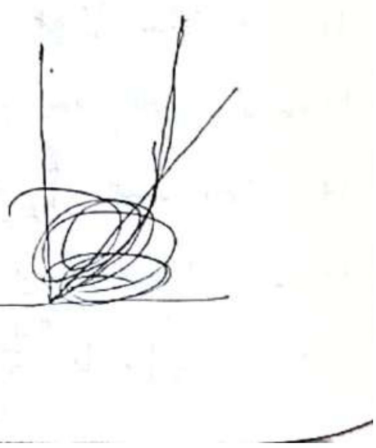
$= \frac{1}{6}\int_0^1 u^{1/2}(1-u)^{3/2} = \frac{1}{6} B\left(\frac{3}{2}, \frac{5}{2}\right)$

$= \frac{1}{6}\,\frac{\Gamma(3/2)\,\Gamma(5/2)}{\Gamma 4} = \frac{\frac{1}{2}\Gamma(1/2)\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}{36} = \frac{\pi}{96}$ as $\Gamma(1/2) = \sqrt{\pi}$

(ii) The required integral

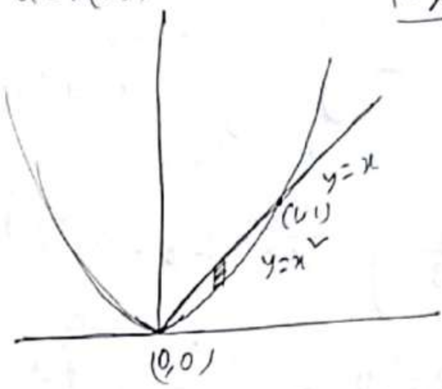
is $= 4\int_0^1\int_0^{\sqrt{1-x^2}} x^2y^2\,dx\,dy = 4\cdot\frac{\pi}{96} = \frac{\pi}{24}$

16. Evaluate $\iint xy(x+y)\,dx\,dy$ over the area bounded by $y=x^2$ and $y=x$



Solution: The required integral I is given by

$$I = \int_0^1 \int_{x^2}^{x} xy(x+y)\,dx\,dy$$



$$= \int_0^1 \left[x^2\frac{y^2}{2} + x\frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right) dx$$

$$= \left[\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$= \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{252 + 168 - 180 - 105}{2520}$$

$$= \frac{135}{2520} = \frac{3}{56}$$

17. Show that $\iint \sqrt{4a^2 - x^2 - y^2}\,dx\,dy$ taken over the upper half within the circle $x^2 + y^2 - 2ax = 0$ is $\frac{4}{9}(3\pi - 4)a^3$

Solution: Here the circle is $(x-a)^2 + y^2 = a^2$

So, the integral $\iint \sqrt{4a^2 - x^2 - y^2}\,dx\,dy$ taken over the upper half within the circle $(x-a)^2 + y^2 = a^2$

Let $x = r\cos\theta$, $y = r\sin\theta$

So, the circle $x^2 + y^2 - 2ax = 0$ becomes

$r^2 - 2ar\cos\theta = 0$ or, $r = 2a\cos\theta$

So, the required integral is

$$I = \int_0^{\pi/2} \int_0^{2a\cos\theta} \sqrt{4a^2 - r^2}\; r\,dr\,d\theta$$ as $\frac{\partial(x,y)}{\partial(r,\theta)} = r = J$ and $r > 0$ except at the pole

Let $4a^2 - r^2 = u^2$ $\therefore -r\,dr = u\,du$

So, $I = -\int_0^{\pi/2}\int_{2a}^{2a\sin\theta} u.u\,du\,d\theta = -\int_0^{\frac{\pi}{2}}\left[\frac{u^3}{3}\right]_{2a}^{2a\sin\theta} d\theta$

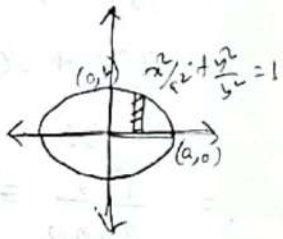
$= -\frac{1}{3}\int_0^{\pi/2}\left(8a^3\sin^3\theta - 8a^3\right)d\theta$

$= 8a^3.\frac{\pi}{2} - \frac{2a^3}{3}\int_0^{\pi/2} 4\sin^3\theta\,d\theta$

$= 8a^3\frac{\pi}{2} + \frac{2a^3}{3}$

18. Show that $\iint\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy = \frac{1}{8}\pi ab$ over the positive quadrant within the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution: The required integral I is given by



$I = \int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy$

$= \int_0^a\left[y-\frac{x^2y}{a^2}-\frac{y^3}{3b^2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$

$= \int_0^a\left[\frac{b}{a}\sqrt{a^2-x^2} - \frac{x^2}{a^2}\cdot\frac{b}{a}\sqrt{a^2-x^2} - \frac{1}{3b^2}\cdot\frac{b^3}{a^3}\left(a^2-x^2\right)^{3/2}\right]dx$

$= \frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx - \frac{b}{a^3}\int_0^a x^2\sqrt{a^2-x^2}\,dx - \frac{b}{3a^3}\int_0^a\left(a^2-x^2\right)^{3/2}dx$

$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a - \frac{b}{a^3}\int_0^a x^2\sqrt{a^2-x^2}\,dx - \frac{b}{3a^3}\int_0^a\left(a^2-x^2\right)^{3/2}dx$

Let $x = a\sin\theta$. So, $dx = a\cos\theta\,d\theta$

x	0	a
θ	0	$\pi/2$

So, $I = \frac{b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} - \frac{b}{a^3}\int_0^{\pi/2}a^2\sin^2\theta.a^2\cos^2\theta\,d\theta - \frac{b}{3a^3}\int_0^{\pi/2}a^3\cos^3\theta.a\cos\theta\,d\theta$