$$= \frac{1}{4}\pi ab - \frac{ab}{4}\int_0^{\pi/2}\sin^2 2\theta\, d\theta - \frac{ab}{3}\cdot\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= \frac{1}{4}\pi ab - \frac{ab}{8}\cdot\int_0^{\pi/2}(1-\cos 4\theta)\,d\theta - \frac{ab}{12}\int_0^{\pi/2}(1+\cos 2\theta)^2 d\theta$$

$$= \frac{1}{4}\pi ab - \frac{ab}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} - \frac{ab}{12}\int_0^{\pi/2}\left[1+2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right]d\theta$$

$$= \frac{1}{4}\pi ab - \frac{1}{16}\pi ab - \frac{ab}{12}\left[\theta + \sin 2\theta + \frac{1}{2}\theta + \frac{\sin 4\theta}{8}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\pi ab - \frac{1}{16}\pi ab - \frac{ab}{12}\times\frac{3}{2}\frac{\pi}{2}$$

$$= \frac{\pi ab}{4} - \frac{\pi ab}{16} - \frac{\pi ab}{16}$$

$$= \frac{4\pi ab - \pi ab - \pi ab}{16} = \frac{2\pi ab}{16} = \frac{1}{8}\pi ab$$

Exercises: 19. Verify that

$\iint y^2\sqrt{a^2-x^2}\,dx\,dy$ extended over the disc $x^2+y^2\le a^2$ is $\frac{32}{45}a^5$.

20. Show that $\iint \frac{x^3+y^3-3xy(x^2+y^2)}{(x^2+y^2)^{3/2}}\,dx\,dy$ over the disc $x^2+y^2\le 1$ is zero.

21. Show that

(i) $\iint y\,dx\,dy = \frac{4}{3}ab^2$ over the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(ii) $\iint x^2y\,dx\,dy = \frac{1}{15}a^3b^2$ taken over the positive quadrant ~~of~~ within the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

UNIT 3 Linear Programming

Motivation of Linear Programming Problem :

In the time of Second World War, an interesting subjects, Known as Operational Research (in short, Called OR), was first developed. Its primary objectives ~~was~~ was to study the strategies and tactics used in the military science so that the limited resources can be utilised most effectively to win the battle. After the end of the war, the success of this military science attracted the attention of the industrialists who were also seeking ~~soon~~ some solutions of the problem that what method should be adopted in their industries to make the total cost minimum or the total profit maximum.

The problems which deal with the best allocation of the available limited resources to any job which is going to be done, are called an allocation problem or programming problem. One of the simpleat programming problem, called optimization problem, is to maximize or minimize a function depending on some variables, where the variables are subjected to certain constraints. One of the simplest optimization problem is Linear Programming Problem, also Known as LPP. In this type of problem, all the functions of the variables are Linear, i.e., the objective function which is to be optimized and the constraints, i.e., the functions showing the limitations of the available resources, are all Linear. The first mathematical technique to solve such type of problems was an iteration method called simplex method and it was

~~devo~~ developed in 1947 by George B. Dantzig, an American Mathematician. Nowadays, Linear Programming technique is used in almost every sphere of life.

Statement of LPP : In the LPP, the linear expression which is to be optimized (maximized or minimized) is called the objective function. The conditions expressed as simultaneous linear equations or inequations are referred to as constraints. The objective function and the constraints are linear function of certain variables which are called decision variables. The coefficients of the decision variables in the objective function are called price parameters.

So, a general linear programming problem can be stated as follows:

Optimize (i.e., maximize or minimize) the linear objective function

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \qquad \cdots \quad (1)$$

subject to the set of constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\leq \text{ or } = \text{ or } \geq)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\leq \text{ or } = \text{ or } \geq)\ b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ (\leq \text{ or } = \text{ or } \geq)\ b_m \end{array}\right\} \quad (2)$$

and the non-negativity restrictions

$$x_j \geq 0, \quad j = 1, 2, \ldots, n.$$

Where $x_1, x_2, \ldots x_n$ are n decision variables and $c_1, c_2, \ldots, c_n$ are n prices parameters. The a_{ij}'s $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$ are called activity parameters and b_i's $(i=1,2,\ldots,m)$ are known as the requirement parameters of the problem.

In compact form, the LPP is stated as follows:

Optimize $z = \sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j \ (\leq \text{ or } = \text{ or } \geq)\ b_i$, $i = 1, 2, \ldots, m$

and $x_j \geq 0$, $j = 1, 2, \ldots, n$

Slack and Surplus variables

It is known that in an LPP, the set of constraints may involve any of the three signs $\leq$, $=$ or $\geq$.

So, let m constraints of an LPP be of the forms

$$\left.\begin{aligned} &\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, r \\ &\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = r+1, r+2, \ldots, s \\ &\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = s+1, s+2, \ldots, m \end{aligned}\right\}$$

The first r constraints involving $\leq$ signs are converted into equations by ~~introduct~~ introducing some non-negative variables, x_{n+i}, $i = 1, 2, \ldots, r$, in the following way:

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i, \quad i = 1, 2, \ldots, r$$

The variables x_{n+i}, $i = 1, 2, \ldots, r$ are called slack variables.

Similarly, ~~last~~ the next $(s-r)$ constraints involving $\geq$ sign are converted into equations by introducing some non-negative variables x_{n+i}, $i = r+1, r+2, \ldots, s$ in the following way:

$$\sum_{j=1}^{n} a_{ij} x_i - x_{n+i} = b_i, \quad i = r+1, r+2, \ldots, s$$

The variables x_{n+i}, $i = r+1, r+2, \ldots, s$ are known as surplus variables.