Theorem 2 Intersection of two convex sets is also a convex set.

Proof: Let X_1 and X_2 be two convex sets in $\mathbb{R}^n$, and let $X = X_1 \cap X_2$. Let $x, y \in X$. So, $x, y \in X_1$ and $x, y \in X_2$. As X_1 and X_2 are convex sets $\lambda x + (1-\lambda) y \in X_1$ and $\lambda x + (1-\lambda) y \in X_2$ for $0 \le \lambda \le 1$.

So, $\lambda x + (1-\lambda) y \in X_1 \cap X_2 = X$

So, $X_1 \cap X_2$ is a convex set.

Theorem 3 A convex polyhedron in $\mathbb{R}^n$ is a convex set.

Proof Let, X be a convex polyhedron in $\mathbb{R}^n$ generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$. So,

$$X = \left\{ x \in \mathbb{R}^n : x = \sum \lambda_i x_i ,\ \lambda_i \geqslant 0,\ i=1,2,\ldots,n \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

Let $x, y \in X$. Then

$$x = \sum_{i=1}^{n} \lambda_i x_i ,\ \lambda_i \geqslant 0,\ i=1,2,\ldots,n \text{ and } \sum_{i=1}^{n} \lambda_i = 1$$

$$y = \sum_{i=1}^{n} \mu_i x_i ,\ \mu_i \geqslant 0,\ i=1,2,\ldots,n \text{ and } \sum_{i=1}^{n} \mu_i = 1$$

Now for $0 \leqq \lambda \leq 1$

$$\lambda x + (1-\lambda) y = \lambda \left(\sum_{i=1}^{n} \lambda_i x_i \right) + (1-\lambda) \left(\sum_{i=1}^{n} \mu_i x_i \right)$$

$$= \sum_{i=1}^{n} \left(\lambda \lambda_i + (1-\lambda) \mu_i \right) x_i = \sum_{i=1}^{n} d_i x_i$$

where $d_i = \lambda \lambda_i + (1-\lambda) \mu_i$, $i=1,2,\ldots n$

As $0 \le \lambda \le 1$, $\lambda_i \geqslant 0$, $\mu_i \geqslant 0$ So, $d_i \geqslant 0$ for $i=1,2,\ldots,n$

Now $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n}(\lambda\lambda_i + (1-\lambda)\mu_i) = \lambda\sum_{i=1}^{n}\lambda_i + (1-\lambda)\sum_{i=1}^{n}\mu_i$

$= \lambda + (1-\lambda)$ as $\sum_{i=1}^{n}\lambda_i = 1$ and $\sum_{i=1}^{n}\mu_i = 1$

$= 1$

So, $\lambda x + (1-\lambda)y \in X$

Hence X is a convex set.

<u>Basic Solution</u>

For a system of linear equation $Ax = b$ of m equations in n unknowns $(n > m)$ where $A = (a_{ij})_{m\times n}$
$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$, if rank of $A = m$, then there exists at least one submatrix B of order m of A which is invertible. Put all the variables in the system zero which are not related to the columns of B. Then the solution of the resulting system is called a basic solution of the system $Ax = b$. We may find ${}^nC_m$ submatrix B of order m. If B is invertible, we get a corresponding basic solution. So, the system $Ax = b$ ~~at most~~ has at most ${}^nC_m$ solutions.
~~In~~ In a basic solution, the variables corresponding to the columns of the submatrix B are called basic variable and other variables which are zero, called non-basic variables.

If in a basic solution, all the basic variables are non-negative, then it is called a basic feasible solution. In short, it is written as ~~BF~~ BFS.
If some basic component in a basic feasible solution is zero, then the basic feasible solution is called a degenerate basic feasible solution. Otherwise, it is called a ~~a~~ non-degenerate basic feasible solution.

Example 1 Find all the basic solutions of the system of linear equations

$$x_1 + x_2 + x_3 = 8$$
$$3x_1 + 2x_2 \quad = 18$$

Solution: The system of linear equations can be written as $Ax = b$ where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 8 \\ 18 \end{pmatrix}$$

Here rank of $A = 2$ as $\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2 \neq 0$

We write $A = (\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\alpha_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are the columns of A

Let $B_1 = [\alpha_1, \alpha_2] = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ As $|B_1| = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1 \neq 0$, B_1 gives a Basic solution. We put $x_3 = 0$ in the system $Ax = b$, so the resulting system is

$$x_1 + x_2 = 8$$
$$3x_1 + 2x_2 = 18$$

So, the solution is $x_1 = 2$, $x_2 = 6$

So, $x_1 = 2$, $x_2 = 6$, $x_3 = 0$ or $(2, 6, 0)$ is a basic solution.
It is also a basic feasible solution as $x_1, x_2 \geq 0$

Let $B_2 = [\alpha_1, \alpha_3] = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}$. As $|B_2| = \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = -3 \neq 0$, B_2 as basic gives a basic solution. We put $x_2 = 0$, in the system $Ax = b$, so the resulting system is

$$x_1 + x_3 = 8$$
$$3x_1 = 18$$

So, $x_1 = 6$, $x_3 = 2$

So, $x_1 = 6$, $x_2 = 0$, $x_3 = 2$ or $(6, 0, 2)$ is a basic solution.
It is also a basic feasible solution.

Let $B_3 = [\alpha_2, \alpha_3] = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$. As $|B_3| = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2 \neq 0$, B_3 gives another basic solution. We put $x_1 = 0$, in the system $Ax = b$, so the resulting system is

$$x_2 + x_3 = 8$$
$$2x_2 = 18$$

So, $x_2 = 9$, $x_3 = -1$

So, another basic solution is $x_1 = 0$, $x_2 = 9$ and $x_3 = -1$ or $(0, 9, -1)$. As $x_3 < 0$, So this is not a basic feasible solution.

Exercise 1: Find all the basic feasible solution of the equation

$$x_1 + x_2 + x_3 = 5$$
$$2x_1 + 2x_2 + x_3 = 7$$