Theorem 4. The set of all feasible solution of an LPP is a convex set.

Proof: Let X be the set of all feasible solutions of an LPP is a convex set whose constraints are

$$Ax = b, \quad x \geq 0$$

Now if $x$ and $y$ be two feasible solution, that is, $x, y \in X$. Then $Ax = b$ and $Ay = b$, $x \geq 0$, $y \geq 0$

now $A(\lambda x + (1-\lambda)y) = \lambda(Ax) + (1-\lambda)Ay = \lambda b + (1-\lambda)b = b$ for $0 \leq \lambda \leq 1$

Also, $\lambda x + (1-\lambda)y \geq 0$ as $0 \leq \lambda \leq 1$, $x \geq 0$, $y \geq 0$

So, $\lambda x + (1-\lambda)y \in X$ for $0 \leq \lambda \leq 1$

So, the set of all feasible solutions is a convex set.

Theorem 5 If any LPP admits of an optimal solution, the objective function assume a its optimal value at an extreme point of the convex set of the set of all feasible solution

Proof: Since in any LPP, a minimization problem can be converted to a maximization problem, so this theorem is established for maximization problem only.

Let $x_1, x_2, \ldots, x_k$ be a finite set of extreme points of the convex set of all feasible solutions. Now, let $y$ be an optimal solution of the LPP and if it can be shown that $y$ is an extreme point of the convex set of all ~~fea~~ feasible solutions, then the theorem is proved.

If possible, let $y$ be an optimal solution but not an extreme point. So, $Z_m = cy$ is the optimal value of the objective function $z$. Since, by assumption, $y$ is ~~a~~ not an extreme point, so it can expressed as a convex combination of extreme points.

$$\text{So,} \quad y = \sum_{i=1}^{k} \lambda_i x_i, \ \lambda_i \geq 0, \ i=1,\dots,k \ \text{and} \ \sum_{i=1}^{k} \lambda_i = 1$$

$$\text{So,} \quad Z_m = cy = c\left(\sum_{i=1}^{k} \lambda_i x_i\right) = \sum_{i=1}^{k} \lambda_i (cx_i)$$

Now if $cx_p = \max_i (cx_i)$, where $x_p$ is an extreme point, then replacing each $cx_i$ by $cx_p$, we have

$$Z_m \leq \sum \lambda_i (cx_p) = cx_p \sum_{i=1}^{k} \lambda_i = cx_p = Z_p \text{ (say)}$$

Then $Z_m \leq Z_p$ which contradicts the assumption that ~~$y$ is~~ $Z_m$ is optimal value of the ~~of~~ objective function. Hence $y$ must be an extreme point.

**Theorem 6** A basic feasible solution (BFS) to an LPP corresponds to an extreme point of the convex set of feasible solutions.

**Proof:** Let for the system of following equations

$$Ax = b, \ x \geq 0 \quad \text{where } A = (a_{ij})_{m \times n} \ (m < n) \qquad \dots (1)$$

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$, there are $m$ linearly independent column vectors of A, say,

$\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m = b \quad \cdots\cdots (1)$$

now let $x = [x_1, x_2, \ldots, x_m, 0, 0, \ldots, 0]$ with $(n-m)$ zeros be a basic feasible solution of (1) and if possible, let $x$ be not an extreme point of the convex set $X$ of the feasible solutions of (1). Then $x$ can be expressed as a convex combination to other points $u, v \in X$

i.e., $x = \lambda u + (1-\lambda) v, \quad 0 < \lambda < 1 \quad \cdots\cdots (3)$

Since all the components of $x$ are non negative (the solutions being feasible) with $(n-m)$ components of it being zero, so, $u$ and $v$ must also have $(n-m)$ zero components.

So, let $u = [u_1, u_2, \ldots, u_m, 0, 0, \ldots, 0]$ and

$$v = [v_1, v_2, \ldots, v_m, 0, 0, \ldots, 0]$$

Again since $u$ and $v$ are two feasible solutions, so by (2),

$$u_1\alpha_1 + u_2\alpha_2 + \cdots + u_m\alpha_m = b$$

$$\text{and } v_1\alpha_1 + v_2\alpha_2 + \cdots + v_m\alpha_m = b$$

And subtracting ~~these~~ these two we get,

$$(u_1 - v_1)\alpha_1 + (u_2 - v_2)\alpha_2 + \cdots + (u_m - v_m)\alpha_m = 0$$

Since $\alpha_1, \alpha_2, \ldots, \alpha_m$ are linearly independent, so,

$$u_1 - v_1 = 0, \; u_2 - v_2 = 0, \; \ldots, \; u_m - v_m = 0$$

or, $u_i = v_i, \; i = 1, 2, \ldots, m$. So, it is not possible to express $x$ as a convex combination of two other distinct points $u$ and $v$ belonging to $X$ and so $x$ is an extreme point. Hence the result.

Statement of Fundamental theorem of an LPP:

If any linear ~~prob~~ programming problem

Maximize $z = cx$

subject to $Ax = b$, $x \geq 0$ where

$A = (a_{ij})_{m \times n}$ $(m < n)$ and $c = (c_1, c_2, \ldots, c_n)$

$x = [x_1, x_2, \ldots, x_n]$, $b = [b_1, b_2, \ldots, b_m]$

admits an optimal solution, then at least one basic feasible solution must be optimal.

Reduction of a feasible solution (FS) to a basic feasible solution (BFS).

Theorem 7: If there be a feasible solution of the system $Ax = b$, $x \geq 0$ where $A = (a_{ij})_{m \times n}$ $x = [x_1, x_2, \ldots, x_n]$, $b = [b_1, b_2, \ldots b_m]$ then there is also a basic feasible solution to this system.

Proof: Let $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are column vectors of $A$.

Let $x = [x_1, x_2, \ldots, x_n]$ be a feasible solution of the given system of the equation $Ax = b$, $x \geq 0$

Let, out of these $n$ ~~variables~~ components of $x$, $k$ be non-zero and the remaining $(n-k)$ are zero.