let, without loss of generality, it be assumed that

$x = [x_1, x_2, \ldots, x_k, \underbrace{0, 0, \ldots 0}_{n-k}]$ now since x is a FS,

so $Ax = b$ or, $\sum_{j=1}^{k} x_j \alpha_j = b$ (1)

and $x \geq 0$, i.e., $x_j \geq 0$, for $j = 1, 2, \ldots, k$ and $x_j = 0$, for $j = k+1, \ldots, n$

where now $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the column vectors of A associated with the k non-zero variables of x. If they are linearly independent, then x is a BFS

now, if $\alpha_1, \alpha_2, \ldots \alpha_k$ be linearly dependent then there exists scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ not all zero such that

$$\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \cdots + \lambda_k \alpha_k = 0$$

or, $$\sum_{j=1}^{k} \lambda_j \alpha_j = 0 \quad \ldots \quad (2)$$

Now, if it is assumed that $\lambda_r \neq 0$, then from (2)

$$\alpha_r = -\sum_{\substack{j=1 \\ j \neq r}}^{k} \frac{\lambda_j}{\lambda_r} \alpha_j$$. Substituting this value of α_r

in (1), we have, $$\sum_{\substack{j=1 \\ j \neq r}}^{k} \left(x_j - x_r \frac{\lambda_j}{\lambda_r} \right) \alpha_j = b. \quad \ldots (3)$$

(3) shows that

$$x' = \left[x_1 - x_r \frac{\lambda_1}{\lambda_r}, \ldots, x_{r-1} - x_r \frac{\lambda_{r-1}}{\lambda_r}, 0\ x_{r+1} - x_r \frac{\lambda_{r+1}}{\lambda_r}, \ldots x_k - x_r \frac{\lambda_k}{\lambda_r}, 0, 0, \ldots 0 \right]$$

is also a solution the system $Ax = b$. Here

x' contains at most $(k-1)$ non-zero components

now to make x' an FS, λ_r is to chosen in such a way that $x_j - x_r \frac{\lambda_j}{\lambda_r} \geq 0$, or, $\frac{x_j}{\lambda_j} \geq \frac{x_r}{\lambda_r}$ if $\lambda_j > 0$

or, $\frac{x_j}{\lambda_j} \leq \frac{x_r}{\lambda_r}$ if $\lambda_j < 0$

Now choosing : $\frac{x_r}{\lambda_r}$ such that

$$\frac{x_r}{\lambda_r} = \min_{\lambda_j > 0} \left\{ \frac{x_j}{\lambda_j} \right\} \text{ or, } \frac{x_r}{\lambda_r} = \max_{\lambda_j > 0} \left\{ \frac{x_j}{\lambda_j} \right\}$$

Then for this $\frac{x_r}{\lambda_r}$, x' is a ~~BF~~ ~~BBS~~ FS. If the column vectors associated with the non-zero components of x' be linearly independent then x' is a BFS, otherwise, using the same process discussed above, as a new FS with less number non-zero variables and ultimately a BFS is obtained.

Example) Starting from the feasible solution (1, 2, 4), obtain a basic feasible solution of the system

$$2x_1 + 3x_2 - x_3 = 4$$

$$3x_1 - x_2 + x_3 = 5$$

Solution: Here the system can be written as

$Ax = b$ where $A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$b = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ rank of $A = 2$ as $\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -2 - 9 = -11 \neq 0$

If $\alpha_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $\alpha_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ then

The system can be written as

$$x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = b \quad \text{---} (1)$$

As rank of $A = 2$. So a basic solution will not contain more than two non-zero components

Since $(1, 2, 4)$ is a feasible solution, so from (1)

$$\alpha_1 + 2\alpha_2 + 4\alpha_3 = b \quad \text{---} (2)$$

Since the given ~~f~~ feasible solution contains three non-zero components, so it is not basic

Also $\alpha_1, \alpha_2, \alpha_3$ are ~~linear~~ linearly dependent

(note rank of $A = 2$ implies maximum number of rows or columns That are independent is 2)

So, there exists scalars $\lambda_1, \lambda_2, \lambda_3$ not all zero, such that

$$\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{---} (3)$$

So, $\lambda_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda_2\begin{bmatrix} 3 \\ -1 \end{bmatrix} + \lambda_3\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or, $\begin{bmatrix} 2\lambda_1 + 3\lambda_2 - \lambda_3 \\ 3\lambda_1 - \lambda_2 + \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So, $2\lambda_1 + 3\lambda_2 - \lambda_3 = 0$

and $3\lambda_1 - \lambda_2 + \lambda_3 = 0$

Hence $\dfrac{\lambda_1}{3-1} = \dfrac{\lambda_2}{-3-2} = \dfrac{\lambda_3}{-2-9}$

or, $\dfrac{\lambda_1}{2} = \dfrac{\lambda_2}{-5} = \dfrac{\lambda_3}{-11} = k$ (say)

So, taking $k=1$, $\lambda_1 = 2$, $\lambda_2 = -5$, $\lambda_3 = -11$

So, (3) can be written as

$$2\alpha_1 - 5\alpha_2 - 11\alpha_3 = 0 \qquad \text{——— (4)}$$

now $\dfrac{x_r}{\lambda_r} = \underset{j}{\text{Min}} \left\{ \dfrac{x_j}{\lambda_j},\ \lambda_j > 0 \right\} = \text{Min}\left(\dfrac{x_1}{\lambda_1}\right) = \dfrac{1}{2} = \dfrac{x_1}{\lambda_1}$

So, x_1 is taken as 0 and new x_2 and x_3 are found by the formula

$$\text{new } x_2 = 2\cancel{x_2} - \tfrac{1}{2}(-5) = 2 + \tfrac{5}{2} = \tfrac{9}{2}$$

$$\text{new } x_3 = 4 - \tfrac{1}{2}(-11) = 4 + \tfrac{11}{2} = \tfrac{19}{2}$$

Since α_2 and α_3 are independent as $\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -2 - 9 = -11 \neq 0$

So, the solution $\left(0, \frac{9}{2}, \frac{19}{2}\right)$ is a basic feasible solution.

<u>Alternative method</u>

From (4) $2\alpha_1 = 5\alpha_2 + 11\alpha_3$

or, $\alpha_1 = \frac{5}{2}\alpha_2 + \frac{11}{2}\alpha_3$

Putting the value of α_1 in (2), we have

$$\tfrac{5}{2}\alpha_2 + \tfrac{11}{2}\alpha_3 \quad 2\alpha_2 + 4\alpha_3 = 6$$

or, $\left(\frac{5}{2} + 2\right)\alpha_2 + \left(\frac{11}{2} + 4\right)\alpha_3 = 6$

or, $\frac{9}{2}\alpha_2 + \frac{19}{2}\alpha_3 = 6$

So, $\left(0, \frac{9}{2}, \frac{19}{2}\right)$ is a feasible solution

As α_2, α_3 are linearly independent, so, $\left(0, \frac{9}{2}, \frac{19}{2}\right)$ is a basic feasible solution.