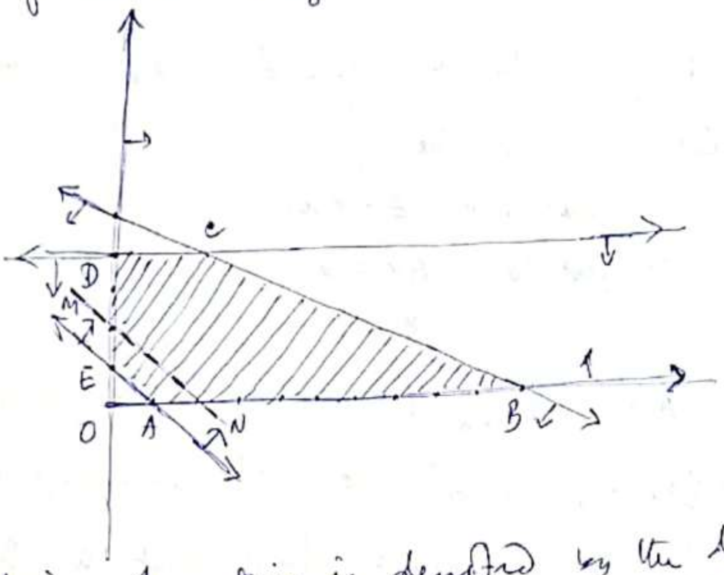
Solution: Changing the inequations to equations, we have

$5x_1 + 10x_2 = 50$ or, $\frac{x_1}{10} + \frac{x_2}{5} = 1$ ... (1)

$x_1 + x_2 = 1$ or, $\frac{x_1}{1} + \frac{x_2}{1} = 1$ ... (2)

$x_2 = 4$ ... (3)

Drawing the graph, the shaded region in the following figure is the feasible region. The region bounded by the polygon ABCDE is the feasible region.



Now giving a particular value to the objective function z, say 2, the corresponding line representing this objective function is denoted by the dotted line MN. Now by moving this line MN parallely, it is seen that this line touches the ~~Q~~ the feasible region at the point B(10, 0) which is farthest from the origin.

So, the maximum value of z occurs at this point

So, $z_{max} = 10 + 0 = 10$ and $x_1 = 10$, $x_2 = 0$ is an optimal solution.

Exercises: 1. Solve graphically the following LPP:

Minimize $z = -x_1 + 2x_2$

subject to $-x_1 + 3x_2 \leq 9$

$x_1 + x_2 \leq 6$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$

~~Exercise~~ 2. Solve graphically the following LPP:

Maximize $Z = 5x_1 + 7x_2$

Subject to $3x_1 + 8x_2 \leq 24$

$x_1 + x_2 \leq 1$

$2x_1 \leq 3$

$x_1, x_2 \geq 0$

Simplex method

As every LPP can be written down as a maximization problem, let the LPP be

$$\text{Maximize } Z = cx$$
$$\text{subject to } Ax = b$$
$$x \geq 0$$

where $A = [a_{ij}]_{m\times n}$ $(m < n)$ and rank of $A = m$

$x = [x_1, x_2, \dots x_n]$ $c = (c_1, c_2, \dots, c_n)$ and $b = [b_1, b_2, \dots, b_m]$

In the simplex, we first find out a BFS of the system $Ax = b$

Then by forming table and making some calculations, we check whether the BFS is an optimal solution or not. If the BFS is optimal, then we get an optimal solution and maximum value of Z. If it is not optimal then we go to the next table and calculations and modify the BFS to a new BFS and check it whether it is optimal or not. In this way we either get an optimal solution or we get some condition from which we can predict that the problem has unbounded solution. As our theory portion is out of syllabus, so we clarify the method by picking a particular problem.

Example 1 Use simplex method to ~~of~~ solve the following LPP:

Maximize $Z = 4x_1 + 10x_2$

subject to $2x_1 + x_2 \le 50$

$2x_1 + 5x_2 \le 100$

$2x_1 + 3x_2 \le 90$

$x_1, x_2 \geqslant 0$

Solution: By using slack variables x_3, x_4 and x_5, the given problem can be put in standard form as

$$\text{Maximize } Z = 4x_1 + 10x_2 + 0.x_3 + 0.x_4 + 0.x_5$$

$$\text{subject to} \quad \begin{aligned} 2x_1 + x_2 + x_3 \qquad\qquad &= 50 \\ 2x_1 + 5x_2 \quad + x_4 \qquad &= 100 \\ 2x_1 + 3x_2 \qquad\quad + x_5 &= 90 \end{aligned}$$

and $x_1, x_2, x_3, x_4, x_5 \geqslant 0$

So, the problem can be written as

$$\text{Max } Z = cx$$

$$\text{subject } Ax = b$$

$$x \geqslant 0$$

where $A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{pmatrix} = (a_1, a_2, a_3, a_4, a_5)$ rank of $A = 3$

$B = I_3 = (a_3, a_4, a_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad b = [50, 100, 90]$

$c = (4, 10, 0, 0, 0)$

$c_B = (c_{B_1}, c_{B_2}, c_{B_3}) \cancel{=(0,0,0)}$ $\quad x_B = [x_{B_1}, x_{B_2}, x_{B_3}] = [x_3, x_4, x_5]$

$= (c_3, c_4, c_5) = (0, 0, 0)$

Here $x_B = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [50, 100, 90] = [50, 100, 90]$

$[\text{As } I^{-1} = I]$

So, the initial BFS is $x_1 = 0$, $x_2 = 0$ $x_3 = 50$, $x_4 = 100$, $x_5 = 90$.

Now we calculate $y_j = B^{-1}a_j = a_j$ since $B^{-1} = I$, $j = 1,2,3,4,5$

Also we calculate $z_1 - c_1 = c_B y_1 - c_1 = [0,0,0]\begin{pmatrix}2\\2\\2\end{pmatrix} - 4 = -4$

Similarly $z_2 - c_2 = c_B y_2 - c_2 = -10$

$z_3 - c_3 = c_B y_3 - c_3 = 0$, $z_4 - c_4 = c_B y_4 - c_4 = 0$,

$z_5 - c_5 = c_B y_5 - c_5 = 0$

If all $z_j - c_j \geq 0$, then our BFS is an optimal solution. As not all $z_j - c_j \geq 0$, this BFS is not an optimal one. So, we have to modify the BFS. Now before that, we say that the calculations are done in a table, called simplex table. So, the 1st simplex table is as follows:

Table - 1

			c_j	4	10	0	0	0	Minimum ratio
c_B	B	x_B	b	y_1	y_2	y_3	y_4	y_5	
0	a_3	x_3	50	2	1	1	0	0	$\frac{50}{1} = 50$
0	a_4	x_4	100	2	5	0	1	0	$\frac{100}{5} = 20$ ⟵
0	a_5	x_5	90	2	3	0	0	1	$\frac{90}{3} = 30$
		$z_j - c_j$		-4	-10 ↑	0	0 ↓	0	

Now the negative most of $z_j - c_j = z_2 - c_2 = -10$, So, the 2nd column is the entering vector in the basis matrix, also called the key column. Now

$$\frac{x_{Br}}{y_{r2}} = \min\left[\frac{x_{B1}}{y_{12}}, \frac{x_{B2}}{y_{22}}, \frac{x_{B3}}{y_{32}}\right]$$ (These y_{i2} are taken which are positive)

$$= \min\left[\frac{50}{1}, \frac{100}{5}, \frac{90}{3}\right]$$

$$= \min[50, 20, 30] = 20 = \frac{x_{B2}}{y_{22}}$$

So, a_4 is the departing vector (as $x_{B2} = x_4$) and the corresponding row, i.e., the ~~secd~~ second row is the key row