Now we state the Fundamental theorem of duality without proof.

Theorem 4 (Fundamental Theorem of duality)

If either the primal Maximize $Z = Cx$ subject to $Ax \le b$, $x \ge 0$ or, the dual, Minimize $W = b^t v$ subject to $A^t v \ge c^t$, $v \ge 0$

has a finite optimal solution, then the other problem will also have a finite optimal solution. Furthermore, the optimal values of the objective functions in both the problem will be same, i.e., $\max Z = \min W$.

We state some more results without proof:

Theorem 5 If the primal (dual) problem has an unbouned solution Then the dual (primal) problem has no feasible solution

Theorem 6 If the dual (primal) problem has no feasible solution then the primal (dual) has unbounded solution.

Transportation Problem

Introduction: Transportation problem is a problem of linear programming. The object of this problem is to transport various amounts of a single homogeneous commodity, initially stored at different sources (or origins), to different destinations in such a way that the cost of transporting is minimum. For this problem, the following informations are to be needed:

(1). Available amount of the commodity at different origins

(2) Amounts demanded at different destinations.

(3) The transportation cost of one unit of commodity from

various origins to various destinations.

Mathematical formulation of transportation problem :

Let, in a transportation problem, there be m origins $O_1, O_2, \ldots, O_m$ with available quantities of commodity $a_1, a_2, \ldots, a_m$ and n destinations $D_1, D_2, \ldots, D_n$ with demands $b_1, b_2, \ldots, b_n$. Let c_{ij}, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, be the transportation cost to transfer one unit of the commodity from the ith origin to the jth destination. Our problem is to determine x_{ij}, the amount of units to be transferred from the ith origin to the jth destination, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, so that the total transportation cost $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$ be minimum. If it is assumed that the total availability is equal to the total demand, i.e., $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, then the transportation problem is called a balanced transportation problem.

This problem can be put in a tabular form as follows :

DESTINATIONS

ORIGINS	D_1	D_2	D_3	$\ldots$	D_n	
O_1	x_{11} c_{11}	x_{12} c_{12}	x_{13} c_{13}	$\ldots$	x_{1n} c_{1n}	a_1
O_2	x_{21} c_{21}	x_{22} c_{22}	x_{23} c_{23}	$\ldots$	x_{2n} c_{2n}	a_2
O_3	x_{31} c_{31}	x_{32} c_{32}	x_{33} c_{33}	$\ldots$	x_{3n} c_{3n}	a_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$		
O_m	x_{m1} c_{m1}	x_{m2} c_{m2}	x_{m3} c_{m3}	$\ldots$	x_{mn} c_{mn}	a_m
	b_1	b_2	b_3		b_n	

Thus a transportation problem can be mathematically put as a linear programming problem as follows:

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{Subject to } \sum_{j=1}^{n} x_{ij} = a_i \,, \quad i=1,2,\dots,m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \,, \quad j=1,2,\dots,n$$

$$x_{ij} \geqslant 0, \text{ for all } i \text{ and } j.$$

If the transportation is a balanced transportation problem then $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

<u>Theorem 1</u> The number of basic variables in a balanced transportation problem is at most $(m+n-1)$

Proof: In a transportation problem with m origins and n destinations, total number of constraints are $m+n$ and they are given by

$$\sum_{j=1}^{n} x_{ij} = a_i \,, \quad i=1,2,\dots,m \quad \dots (1)$$

$$\text{and} \quad \sum_{i=1}^{m} x_{ij} = b_j \,, \quad j=1,2,\dots,n \quad \dots (2)$$

Again, since this a balanced transportation problem, so

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \dots (3)$$

Now from (1), using (3), we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \dots (4)$$

With the help of (4), we will show that one constraint is redundant, i.e., one constraint can be obtained from the constraints. Summing the first $(n-1)$ constraints of (2), we get

$$\sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \quad \ldots \quad (5)$$

Subtracting (5) from (4), we get

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} x_{ij}\right) = \sum_{j=1}^{n} b_j - \sum_{j=1}^{n-1} b_j$$

or, $$\sum_{i=1}^{m} x_{in} = b_n$$ which is nth constraint of (2). Thus, the number of basic variables of a transportation problem is at most $(m+n-1)$.

Note 1. We can prove that the number of basic variables of a transportation problem is exactly $(m+n-1)$.

Note 2. Henceforth, we will write the transportation problem as TP.

Theorem 2 (Existence of a feasible solution)

A necessary and sufficient condition for the existence of a feasible solution to a transportation problem is

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Proof: The condition is necessary:

Let the TP has a feasible solution. Then we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \text{ and}$$

$$\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j \quad \text{and thus } \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The condition is sufficient:

Let $$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = M \text{ (say)}$$