The element in the intersection of key row and key column is called the key element. Here the key element is 5 and is marked by enclosing in a square.

The second table is found from the first table as follows. Since a_2 is the entering vector and a_4 is the departing vector, so x_4 and a_4 is replaced by x_2 and a_2 and the value C_{B_2} is changed from 0 to 10.

Now all elements of second row, i.e., the key row is divided by the key element 5 (y_{22}) and this becomes the second row of the second table. Then with the help of this new row, by row operations, we change the other two rows such that other elements in the key column become zero. For example, as the element in the first row at the key column position is 1, we subtract the new second row from old first row and similarly, we first multiply the new second row by 3 and then subtract it from the third row to get the new third row. In this way the second table is filled. $z_j - c_j$ are calculated similarly. The second table and the calculations of $z_j - c_j$ is shown in Table-2 below:

Table-2

			c_j	4	10	0	0	0	Minimum ratio
C_B	B	x_B	b	y_1	y_2	y_3	y_4	y_5	
0	a_3	x_3	30	$\boxed{8/5}$	0				$\frac{75}{4}$ ←
10	a_2	x_2	20	$2/5$	1	0	$\frac{1}{5}$	0	50
0	a_5	x_5	30	$4/5$	0	0	$-\frac{3}{5}$		$\frac{75}{2}$
$z_j - c_j$				0 ↑	0	0 ↓	2	0	

(See Note 3)

It is seen that all $z_j - c_j \geq 0$. So the new solution is optimal. So, an optimal solution is

$x_1 = 0,\ x_2 = 20,\ x_3 = 30,\ x_4 = 0,\ x_5 = 30$

and $Z_{max} = 10 \times 20 = 200$

So, ~~the~~ an optimal solution of the LPP is

$x_1 = 0$ and $x_2 = 20$ and $Z_{max} = 200$

Note 1. B is called the basis matrix, x_B is called the basic component of the solution, i.e., in the 1st table, x_3, x_4, x_5 are basic variables and $x_B = [x_3, x_4, x_5]$. Other components are called non-basic variables. b is the solution column.

Note 2 If $z_j - c_j \geqslant 0$ for all j in a maximization problem, then the optimality condition is satisfied.

Note 3 At the optimal table, i.e., when all $z_j - c_j \geqslant 0$ for all, if $z_j - c_j = 0$ for some non-basic variable x_j, then the optimal solution is not unique and we can find another optimal solution by taking the column vector corresponding to this variable as entering vector. In table 2 of Example 1, we see all $z_j - c_j \geqslant 0$ and $z_1 - c_1 = 0$ for the non-basic variable x_1. So, the optimal solution is not unique. Now using a_1 as entering vector, we compute the third simplex table as usual and is shown as follows

Table-3

see Table 2 for entering & departing vector etc.

			c_j	4	10	0	0	0
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
4	a_1	x_1	$\frac{75}{4}$	1	0	$\frac{5}{8}$	$-\frac{1}{8}$	0
10	a_2	x_2	$\frac{25}{2}$	0	1	$-\frac{1}{4}$	$\frac{1}{4}$	0
0	a_5	x_5	15	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1
		$z_j - c_j$		0	0	0	2	0

Here $z_j - c_j \geqslant 0$ for all j also.

So an alternative optimal solution is $x_1 = \frac{75}{4}$, $x_2 = \frac{25}{2}$

and $Z_{max} = 4 \times \frac{75}{4} + 10 \times \frac{25}{2} = 200$.

Note 4 It is to be noted if some problem has more than one optimal solution then there exists infinite number of optimal solutions. For example, we have seen that $x = (0, 20)$ and $y = (\frac{75}{4}, \frac{25}{2})$ are two optimal solutions. Then $\lambda x + (1-\lambda) y$ is also an optimal solution for any λ such that $0 \le \lambda \le 1$

Let $\lambda = \frac{1}{3}$, then $1-\lambda = 1-\frac{1}{3} = \frac{2}{3}$

$$\text{So, } \frac{1}{3}x + \frac{2}{3}y = \frac{1}{3}(0, 20) + \frac{2}{3}\left(\frac{75}{4}, \frac{25}{2}\right)$$

$$= \left(0, \frac{20}{3}\right) + \left(\frac{75}{6}, \frac{25}{3}\right)$$

$$= \left(\frac{75}{6}, 15\right)$$

So, $x_1 = \frac{75}{6}$ and $x_2 = 15$ is also an optimal solution

and $Z_{max} = \frac{75}{6} \times 4 + 15 \times 10 = 50 + 150 = 200$

Note We always try to find the basis matrix B initially as identity matrix I, so that $B^{-1} = I^{-1} = I$. Slack variable gives a column of the identity matrix. If there is lack of column for identity matrix, we, forcefully introduce artificial variables to get the columns of a identity matrix and as a penalty, we take $-M$ as the coefficient of the artificial variable in the objective function, where M is a large positive real number. Then we do our calculations as usual only remembering that M is a large positive real numbers. If artificial variables present at the table at zero level or absent at the optimal table, then we get an optimal solution. If artificial variable present at the table at non-zero level, then the problem has no feasible solution. (This method is called Charne's Big-M method)

Exercise 1 Solve using simplex method,

$$\text{Maximize } z = 6x_1 - 2x_2$$

subject to $2x_1 - x_2 \leq 2$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Example 2 Solve the following LPP by simplex method

$$\text{Minimize } z = 3x_1 - 2x_2$$

subject to $x_1 - x_2 \leq 1$

$3x_1 - 2x_2 \leq 6$

$x_1, x_2 \geq 0$

Solution: At first, the ~~maximization~~ minimization problem is converted to maximization problem as follows:

As we know ~~Max Z~~ = Min z = − Max (−z) = − Max z′, where z′ = −z, so we take the problem as

$$\text{Maximize } z' = -3x_1 + 2x_2$$

subject to $x_1 - x_2 \leq 1$

$3x_1 - 2x_2 \leq 6$

$x_1, x_2 \geq 0$

Now using slack variables x_3 and x_4, the LPP can be put to standard form as,

$$\text{Maximize } z' = -3x_1 + 2x_2 + 0 \cdot x_3 + 0 \cdot x_4$$

subject to $x_1 - x_2 + x_3 = 1$

$3x_1 - 2x_2 + x_4 = 6$

$x_1, x_2, x_3, x_4 \geq 0$

So, our simplex table is

		c_j		-3	2	0	0
c_B	B	x_B	b	y_1	y_2	y_3	y_4
0	a_3	x_3	1	1	-1	1	0
0	a_4	x_4	6	3	-2	0	1
	$z_j - c_j$			3	-2 ↑	0	0

Here $z_2 - c_2 < 0$