where $A = (a_{ij})_{m\times n}$ $c = (c_1, c_2, \ldots, c_n)$, $b = [b_1, b_2, \ldots, b_m]$, $x = [x_1, x_2, \ldots, x_n]$ $v = [v_1, v_2, \ldots, v_m]$ and A^t, b^t and c^t are the corresponding ~~most~~ transpose matrices (first bracket ( ) means row vector and third bracket means column ~~as~~ ~~most~~ vector as mentioned earlier)

So, in details, if the primal problem be

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$
$$x_1, x_2, \ldots, x_n \ge 0$$

then its dual problem is

$$\text{Minimize } W = b_1v_1 + b_2v_2 + \cdots + b_mv_m$$

subject to

$$a_{11}v_1 + a_{21}v_1 + \cdots + a_{m1}v_m \ge c_1$$
$$a_{12}v_1 + a_{22}v_2 + \cdots + a_{m2}v_m \ge c_2$$
$$\vdots$$
$$a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{mn}v_m \ge c_n$$
$$v_1, v_2, \ldots, v_m \ge 0$$

Notes: In the primal-dual problem, it is observed that

(i) The number of variables in the dual is the same as the number of constraints in the primal and vice-versa.

(ii) The elements of the requirement vectors and prices in the objective functions are interchanged in these two problems.

(iii) If the primal (dual) problem is a maximization then its dual (primal) is a minimization problem.

(iv) The variables in both the problems are non-negative

(v) The columns of the coefficient matrix are the activity vectors of the primal problem and on the other hand the rows of the coefficient matrix are the ~~activitist~~ activity vectors of the dual problems.

Example 1 Write the dual of the following LPP

$$\text{Maximize } Z = x_1 - x_2 + 3x_3$$

subject

$$x_1 + x_2 + x_3 \leq 10$$
$$2x_1 \quad - x_3 \leq 2$$
$$2x_1 - 2x_2 + 3x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0$$

Solution: The given primal problem is already in standard form. So the dual problem is

$$\text{Minimize } W = 10v_1 + 2v_2 + 6v_3$$

subject to

$$v_1 + 2v_2 + 2v_3 \geq 1$$
$$v_1 \quad - 2v_3 \geq -1$$
$$v_1 - v_2 + 3v_3 \geq 3$$
$$v_1, v_2, v_3 \geq 0$$

Example 2 Write down the dual of the following problem:

$$\text{Maximize } Z = 3x_1 + 2x_2 + x_3$$

subject to

$$x_1 - 3x_2 + 2x_3 \leq 5$$
$$x_1 + 5x_2 + x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

Solution: The standard primal problem will be

Maximize $Z = 3x_1 + 2x_2 + x_3$

subject to $x_1 - 3x_2 + 2x_3 \leq 5$

$-x_1 - 5x_2 - x_3 \leq -8$

$x_1, x_2, x_3 \geq 0$

So, the dual is

Minimize $W = 5v_1 - 8v_2$

subject to $v_1 - v_2 \geq 3$

$-3v_1 - 5v_2 \geq 2$

$2v_1 - 3v_2 \geq 1$

$v_1, v_2 \geq 0$

Example 3 Find the dual of the following primal problem:

Minimize $Z = 3x_1 - 2x_2$

subject to $2x_1 + x_2 \leq 1$

$-x_1 + 3x_2 \geq 4$

$x_1, x_2 \geq 0$

Solution: The given primal problem can be put in standard form as follows:

Maximize $Z' = -3x_1 + 2x_2$ where $\text{Min } Z = -\text{Max}(Z')$

subject to $2x_1 + x_2 \leq 1$

$x_1 - 3x_2 \leq -4$

$x_1, x_2 \geq 0$

So, the dual is

Minimize $W' = v_1 - 4v_2$

subject to $2v_1 + v_2 \geq -3$

$v_1 - 3v_2 \geq 2$

$v_1, v_2 \geq 0$

which is equivalent to the problem

Maximize $W = -v_1 + 4v_2$ where $\text{Max } \cancel{W} \;\; W = -W'$

$\text{Min } W' = -\text{Max}(W')$

subject to $2v_1 + v_2 \geq -3$

$v_1 - 3v_2 \geq 2$

$v_1, v_2 \geq 0$

Example 4 Given the LPP

Maximize $Z = 2x_1 + 3x_2 + 4x_3$

Subject to $x_1 - 5x_2 + 3x_3 = 7$

$2x_1 - 5x_2 \leq 3$

$3x_2 - x_3 \geq 5$

$x_1, x_2 \geq 0$ and x_3 is unrestricted in sign.

Solution: We first find the standard form of the primal. The equation $x_1 - 5x_2 + 3x_3 = 7$ can be replace by two inequalities as

$$x_1 - 5x_2 + 3x_3 \leq 7$$
$$\text{and } x_1 - 5x_2 + 3x_3 \geq 7$$

[as $a = 7$ is equivalent to $a \leq 7$ and $a \geq 7$]

or, This can be written as

$$x_1 - 5x_2 + 3x_3 \leq 7$$
$$\text{and } -x_1 + 5x_2 - 3x_3 \leq -7$$

Also as x_3 is unrestricted in ~~sing~~ sign, x_3 can be written as $x_3 = x_3' - x_3''$ where $x_3' \geq 0$ and $x_3'' \geq 0$

So, the given primal problem can be written in the standard form as follows: Maximize : $Z = 2x_1 + 3x_2 + 4x_3' - 4x_3''$

subject to

$$x_1 - 5x_2 + 3x_3' - 3x_3'' \leq 7$$
$$-x_1 + 5x_2 - 3x_3' + 3x_3'' \leq -7$$
$$2x_1 - 5x_2 \leq 3$$
$$-3x_2 + x_3' - x_3'' \leq -5$$
$$x_1, x_2, x_3', x_3'' \geq 0$$