So, the dual is

$$\text{Minimize } W = 7v_1' - 7v_1'' + 3v_2 - 5v_3$$

subject to
$$v_1' - v_1'' + 2v_2 \geq 2$$
$$-5v_1' + 5v_1'' - 5v_2 - 3v_3 \geq 3$$
$$3v_1' - 3v_1'' + v_3 \geq 4$$
$$-3v_1' + 3v_1'' - v_3 \geq -4$$
$$v_1', v_1'', v_2, v_3 \geq 0$$

Now writing $v_1 = v_1' - v_1''$ where v_1 is unrestricted in sign

and also the last two inequalities become $3v_1 + v_3 \geq 4$

and $-3v_1' - v_3 \leq -4$ or $3v_1 + v_3 \geq 4$

as they are equivalent to $3v_1 + v_3 = 4$

So, the dual becomes

$$\text{Minimize } W = 7v_1 + 3v_2 - 5v_3$$

Subject to
$$v_1 + 2v_2 \geq 2$$
$$-5v_1 - 5v_2 - 3v_3 \geq 3$$
$$3v_1 + v_3 = 4$$

$v_2, v_3 \geq 0$ and v_1 is unrestricted in sign

Note: If there is an unrestricted variable in primal then there is a equality constraint in the dual and if there is a equality constraint in the dual there is an unrestricted variable in the primal.

Exercises: 1. Find the dual of LPP

$$\text{Maximize } Z = 4x_1 + 3x_2$$

subject to $x_1 + x_2 \leq 5$

$2x_1 - 3x_2 \leq 2$

$x_1, x_2 \geq 0$

2. Find the dual of the LPP

$$\text{Minimize } Z = 3x_1 + 2x_2$$

subject to $2x_1 + 4x_2 \geq 7$

$3x_1 + 2x_2 \geq 11$

$x_1, x_2 \geq 0$

3. Find the dual of the LPP

Maximize $Z = 2x_1 - 6x_2$

subject to $x_1 - 3x_2 \leq 6$

$2x_1 + 4x_2 \geq 8$

$x_1 - 3x_2 \geq -6$

$x_1, x_2 \geq 0$

4. Obtain the dual of the following LPP

Maximize $z = x_1 - x_2 + 3x_3$

subject to $x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$

$2x_2 - x_3 \geq 4$

$x_1, x_2 \geq 0$, x_3 is unrestricted in sign.

Some properties of duality.

Here we take the primal problem as

$$\left.\begin{aligned} &\text{Maximize } Z = cx \\ &\text{subject to } Ax \leq b \\ &\qquad x \geq 0 \end{aligned}\right\} \quad \cdots \quad (1)$$

and the dual problem is

$$\left.\begin{aligned} &\text{Minimize } W = b^t v \\ &\text{subject to } A^t v \geq c^t \\ &\qquad v \geq 0 \end{aligned}\right\} \quad \cdots \quad (2)$$

where $A = [a_{ij}]_{m \times n}$ $c = (c_1, c_2, \ldots, c_n)$ $x = [x_1, x_2, \ldots, x_n]$

$b = [b_1, b_2, \ldots, b_m]$ and $v = [v_1, v_2, \ldots, v_m]$ and A^t, b^t, c^t

are the transpose matrices of A, b and c respectively

Theorem 1 The dual of the dual is primal.

Proof: we know that (2) is the dual of (1). Now we find the dual of (2), we shall first write (2) in the standard primal form, we know ~~Minimum W = Maximum (-W~~

Minimum $W = -$ Maximum $(-W) = -$ Max W' where $W' = -W = -b^t v$

So, (2) is written as

Maximize $W' = -b^t v$

subject to $-A^t v \leq -c^t$

$v \geq 0$

So, the dual can be written as

~~Hence~~ Minimize $z' = (-c^t)^t x$
subject $(-A^t)^t x \geq (-b^t)^t$
$x \geq 0$
$\Bigg\} \quad \cdots \quad (3)$

Now the dual (3) can be written as

$$\text{Minimize } z' = -cx$$
$$\text{subject to } -Ax \geq -b$$
$$x \geq 0$$

But since Minimum $z' = -\text{Max}(-z') = -\text{Max } cx = -\text{Max } z$

and $-Ax \geq -b$ is equivalent $Ax \leq b$

So, this problem ultimately reduces to

$$\text{Maximize } z = cx$$
$$\text{subject to } Ax \leq b$$
$$x \geq 0$$

which is same as (1)

Hence the theorem follows.

Theorem 2 (Weak duality Theorem) : If x_0 be any feasible solution to the primal problem Maximize $z = cx$
subject to $Ax \leq b$
$x \geq 0$

and v_0 be any feasible solution to the dual problem
Minimize $W = b^t v$
subject to $A^t v \geq c^t$
$v \geq 0$

then $cx_0 \leq b^t v_0$

Proof: As v_0 is a feasible solution of the dual, we have,

$$A^t v_0 \geq c^t \quad \text{or, } (A^t v_0)^t \geq (c^t)^t \quad \text{or } v_0^t A \geq c \quad \cdots (1)$$

Postmultiplying (1) by x_0, we have

$$(v_0^t A) x_0 \geq c x_0 \quad [\because x_0 \geq 0]$$

$$\text{or, } v_0^t (A x_0) \geq c x_0$$

or, $v_0^t b \geq cx_0$ $[\because Ax_0 \leq b]$

or, $b^t v_0 \geq cx_0$ (As $v_0^t b$ is a scalar)

or, $cx_0 \leq b^t v_0$.

Hence the theorem is proved.

Theorem 3 If x_0 is a feasible solution to the primal problem

$$\text{Maximize } z = cx$$
$$\text{Subject to } Ax \leq b$$
$$x \geq 0$$

and v_0 is a feasible solution to the dual problem

$$\text{Minimize } W = b^t v$$
$$\text{subject to } A^t v \geq c^t$$
$$v \geq 0$$

and $cx_0 = b^t v_0$, then x_0 and v_0 are the optimal feasible solutions of the primal and dual respectively.

Proof: Let x and v be any two feasible solutions of the primal and dual respectively. Then by Theorem 2

$$cx \leq b^t v$$

Then $cx \leq b^t v_0$ (As v_0 is a feasible solution of the dual)

or, $cx \leq b^t v_0 = cx_0$

or, $cx \leq cx_0$

So, x_0 is an optimal feasible solution of the primal and $\max z = cx_0$

In the same way $cx_0 \leq b^t v$ (as x_0 is a feasible solution of the primal)

So, $b^t v_0 = cx_0 \leq b^t v$

or, $b^t v_0 \leq b^t v$

So, v_0 is an optimal feasible solution of the dual and $\min W = b^t v_0$. Hence the theorem.

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