Some more exercises :

1. Let G be an infinite cyclic group. Then prove that G has one non-trivial isomorphism.

Proof: Since G is isomorphic to $\mathbb{Z}$, it is sufficient to determine all automorphisms of $\mathbb{Z}$. Let $\phi: \mathbb{Z} \to \mathbb{Z}$ be an automorphism of $\mathbb{Z}$. Since 1 generates $\mathbb{Z}$ and ϕ is an automorphism, $\phi(1)$ generates $\mathbb{Z}$. Since the only generators of $\mathbb{Z}$ are ± 1 (Problem 17, Page-62), $\phi(1) = \pm 1$. If $\phi(1) = 1$ then $\phi(n) = n$ for each $n \in \mathbb{Z}$. So, ϕ is the trivial automorphism. If $\phi(1) = -1$ then $\phi(n) = -n$, for each $n \in \mathbb{Z}$. So this is the only non-trivial automorphism.

2. Let G be a group. Consider the sets

$$S = \{aba^{-1}b^{-1} : a, b \in G\}$$

and $K = \{s_1 s_2 \cdots s_m : s_1, s_2, \ldots, s_m \in S \text{ and } m \in \mathbb{N}\}$ ($\mathbb{N}$ is the set of natural numbers). Then prove that K is a subgroup of G.

Proof: Let $a, b \in K$. So, $a = s_1 s_2 \cdots s_m$, $b = s_1' s_2' \cdots s_n'$ where $s_1, s_2, \ldots, s_m, s_1', s_2', \ldots, s_n' \in S$. Then

$$ab^{-1} = (s_1 s_2 \cdots s_m)(s_1' s_2' \cdots s_n')^{-1}$$
$$= s_1 s_2 \cdots s_m s_n'^{-1} s_{n-1}'^{-1} \cdots s_1'^{-1}.$$

Now, since $s_1, \ldots, s_m \in S$ and also $s_1'^{-1}, s_2'^{-1}, \ldots, s_n'^{-1} \in S$, so $ab^{-1} \in K$. Also $K \neq \phi$ as $e \in K$, e is the identity in G. So K is a subgroup of G.

K is called the commutator subgroup of G.

3. Show that the commutator subgroup K is a normal subgroup

Proof: Let $a \in K$ and $x \in G$. Then $xax^{-1} = (xax^{-1}a^{-1})a$

Now $xax^{-1}a^{-1} \in K$ and $a \in K$. Since K is a subgroup of G, so $xax^{-1} = (xax^{-1}a^{-1})a \in K$. Hence K is a normal subgroup of G.

4. Prove that G is an abelian group if and only if the commutator subgroup of G is the trivial subgroup.

Proof. Let G be abelian. Let $a, b \in G$. Then $aba^{-1}b^{-1} = abb^{-1}a^{-1} = aea^{-1} = aa^{-1} = e$, e is the identity element in G. So the commutator subgroup whose elements will be finite product of e's is the trivial subgroup.

Conversely, let the commutator subgroup be the trivial subgroup. Then for $a, b \in G$, $aba^{-1}b^{-1}$ belongs to the commutator subgroup.

So $aba^{-1}b^{-1} = e$, So, $ab = ba$. Hence G is abelian.

5. If K is a normal subgroup of G and H is any subgroup of G, show that $H \cap K$ is a normal subgroup of H.

Proof: As K and H are subgroups of G. So, $H \cap K$ is subgroup of G. As $H \cap K \subset H$, So, $H \cap K$ is a subgroup of H. To show that $H \cap K$ is a normal subgroup of H, let

$a \in H \cap K$ and $x \in H$. As K is normal subgroup of G and $a \in K$. So $yay^{-1} \in K$ for all $y \in G$.

Hence $xax^{-1} \in K$. Now $a \in H$ and $x \in H \Rightarrow xa$ and $x^{-1} \in H$ as H is a subgroup and so also $xax^{-1} \in H$.

So, $xax^{-1} \in H \cap K$. Hence $H \cap K$ is a normal subgroup of H.

6. Let $\text{Aut}\, G$ be the group of all automorphisms of G. Show that the set of all inner automorphisms $I(G)$ forms a normal subgroup of $\text{Aut}\, G$.

Proof: Do it yourself.

7. Let $\theta: G \to G'$ be a homomorphism of G into G' with G' abelian. Let H be a subgroup of G containing $\text{Ker}\,\theta$. Show that H is normal in G.

Proof: Let $h \in H$ and $x \in G$. ~~Then consider~~ Then

$$\theta(xhx^{-1}h^{-1}) = \theta(x)\theta(h)\theta(x^{-1})\theta(h^{-1})$$

$$= \theta(x)\theta(x^{-1})\theta(h)\theta(h^{-1}) \quad [\text{As } G' \text{ is abelian}]$$

$$= \theta(xx^{-1})\theta(hh^{-1})$$

$$= \theta(e)\theta(e) \quad [e \text{ is the identity element in } G]$$

$$= e'\cdot e'$$

$$= e' \quad [e' \text{ is the identity element in } G']$$

So, $xhx^{-1}h^{-1} \in \ker\theta \subset H$

So, $xhx^{-1}h^{-1} \in H$ and also $h \in H$ and as H is a subgroup of G, $(xhx^{-1}h^{-1})h \in H$ or $xhx^{-1} \in H$. So, H is a normal subgroup of G.

8. Let G be a group in which $(ab)^3 = a^3b^3$ and $(ab)^5 = a^5b^5$ for all $a, b \in G$, prove that the group is abelian.

Proof: Let $a, b \in G$

Then $(ab)^3 = a^3b^3$

or, $(ab)(ab)(ab) = a^3b^3$

or, $a(ba)^2b = a^3b^3$ ——— (1)

or, $(ba)^2 = a^2b^2$ (by cancellation law)

Similarly as $(ab)^5 = a^5b^5$,

we have $(ba)^4 = a^4b^4$ ——— (2)

From (2), $(ba)^2(ba)^2 = a^4b^4$

or, $(a^2b^2)(a^2b^2) = a^4b^4$ (from (1))

or, $b^2a^2 = a^2b^2$ (cancellation law)

or, $b^2a^2 = (ba)^2$ (from (1))

or, $b^2a^2 = (ba)(ba)$ (by Cancellation law)

or, $ba = ab$

So, G is abelian as $ab = ba$, for all $a, b \in G$.

9. In a group G, $a^2b^2 = b^2a^2$ and $a^3b^3 = b^3a^3$ hold for all $a, b \in G$, prove that the group G is abelian.

Proof: It is the problem 3 of Page-54 when $m=2$ and $n=3$.

10. In a group G, a and b are distinct elements of order 2.
(i) If a and b commute, prove that $o(ab) = 2$
(ii) If a and b do not commute
(iii) Deduce that a group G can not contain exactly two elements of order 2.

Proof: (i) Let a, b commute. Then $ab = ba$

Now, $o(a) = o(b) = 2$, So, $a^2 = e$, $b^2 = e$, e is the identity in G.

So, $a = a^{-1}$, $b = b^{-1}$

Now $ab = e \Rightarrow b = a^{-1} = a$, a contradiction that a, b are distinct. So, $ab = ba \neq e$ Similarly $ab \neq a$, $ab \neq b$

Now $(ab)^2 = (ab)(ab) = a(ba)b = a(ab)b = a^2b^2 = e$

So, $o(ab) = 2$

(ii) Here $ab \neq ba$ $aba^{-1} = e \Rightarrow ab = a \Rightarrow b = e$, a contradiction as $o(b) = 2$. So, $aba^{-1} \neq e$ Also $aba^{-1} = a \Rightarrow ab = a^2 = e \Rightarrow a = b^{-1} = b$, a contradiction. So, $aba^{-1} \neq a$. Again, $aba^{-1} = b \Rightarrow ab = ba$, a contradiction that a, b do not commute. So, $aba^{-1} \neq b$

Now $(aba^{-1})^2 = (aba^{-1})(aba^{-1}) = ab^2a^{-1} = aea^{-1} = aa^{-1} = e$

So, $o(aba^{-1})^2 = e$.

(iii) In both cases below, we see that if $\exists$ two distinct elements a and b of order 2 $\exists$ another distinct element (either ab or aba^{-1}) of order 2. Hence the result.

11. If b an element of a group G and $o(b) = 20$, find the order of (i) b^6 (ii) b^8 (iii) b^{15}

Solution : (i) $o(b^6) = \frac{20}{\gcd(6,20)} = \frac{20}{2} = 10$. So $o(b^6) = 10$

(ii) $o(b^8) = \frac{20}{\gcd(8,20)} = \frac{20}{4} = 5$. So, $o(b^8) = 5$

(iii) $o(b^{15}) = \frac{20}{\gcd(15,20)} = \frac{20}{5} = 4$. So, $o(b^{15}) = 4$

12. Let G be a group and $a, b \in G$. If $o(a) = 3$ and $aba^{-1} = b^2$, find $o(b)$ if $b \neq e$.

Solution: Here, $aba^{-1} = b^2 \Rightarrow a^2ba^{-2} = ab^2a^{-1} = (aba^{-1})^2 = b^4$

$\Rightarrow a^3ba^{-3} = ab^4a^{-1} = (aba^{-1})^4 = b^8$

$\Rightarrow b = b^8$ (As $a^3 = e$, e is the identity element in G)

$\Rightarrow b^7 = e$

Let $o(b) = m$. So, m divides 7. So, $m = 1$ or 7

As $b \neq e$, $m \neq 1$ So, $m = 7$

So, $o(b) = 7$

13. Let G be a group and $a, b \in G$. If $a^2 = e$ and $ab^2a = b^3$, prove that $b^5 = e$, e is the identity element in G.

Proof Here $a^2 = e \Rightarrow a = a^{-1}$

$ab^2a = b^3$

$\Rightarrow a^2b^2a^2 = ab^3a$

$\Rightarrow b^2 = ab^3a$ [as $a^2 = e$]

$\Rightarrow b^4 = (ab^3a)(ab^3a) = ab^6a$ [as $a^2 = e$]

$\Rightarrow b^4 = (ab^2a)(ab^2a)(ab^2a)$ [as $a^2 = e$]

$= b^3 \cdot b^3 b^3 = b^9$

$\Rightarrow b^5 = e$