SOLVE IT ⇒

Model Question Paper for CC-4 — (1)

Full marks: 65

1. Choose the correct alternative. Justify your answer. 2X10

(a) The solution of the equation $ax = b$ in the group $S_3$, where $a = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, is

(i) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

(b) Let $H$ and $K$ be two finite subgroups of a group $G$ such that $HK$ is a subgroup and $H \cap K = \{e\}$, $e$ is the identity element in $G$. If $o(H) = 3$, $o(K) = 4$. Then $o(HK)$ is

(i) 6 (ii) 24 (iii) 12 (iv) 36

(c) Two subgroups of $(\mathbb{Z}, +)$ whose unions are subgroups, are

(i) $(2\mathbb{Z}, +)$ and $(3\mathbb{Z}, +)$ (ii) $(3\mathbb{Z}, +)$ and $(4\mathbb{Z}, +)$ (iii) $(4\mathbb{Z}, +)$ and $(7\mathbb{Z}, +)$

(iv) $(3\mathbb{Z}, +)$ and $(6\mathbb{Z}, +)$

(d) Let $G$ be a cyclic group of order 30. Then the number of generators of $G$ are

(i) 8 (ii) 12 (iii) 9 (iv) 7

(e) In a group $G$, $a$ is an element of order 30. $o(a^{18})$ is

(i) 6 (ii) 7 (iii) 5 (iv) 4

(f) An element of order 6 in the group $S_3$ is

(i) $a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ (iv) $S_3$ has no element of order 6.

(g) If $S_6$ is the symmetric group, Then the order of the alternating group $A_6$ is

(i) 320 (ii) 340 (iii) 350 (iv) 360

(h) The elements of order 8 in the group $(\mathbb{Z}_{24}, +)$ are

(i) $\bar{2}, \bar{4}, \bar{7}, \overline{22}$ (ii) $\bar{3}, \bar{6}, \overline{16}, \overline{21}$ (iii) $\bar{2}, \bar{9}, \overline{16}, \overline{21}$ (iv) $\bar{3}, \bar{9}, \overline{15}, \overline{21}$

(i) An example of a non-cyclic group is

(i) $(2\mathbb{Z},+)$ (ii) $(\mathbb{Z},+)$ (iii) $(\mathbb{Q},+)$ (iv) $(3\mathbb{Z},+)$

(j) The centre $Z(S_3)$ of the group $S_3$ has

(i) 2 elements (ii) 3 elements (iii) 1 element (iv) 6 elements

2. Answer one question from (a) and (b), one question from (c) and (d); and one question from (e) and (f)

(a) (i) Give an example of a non-cyclic abelian group 2

(ii) Let G be an infinite cyclic group. ~~generated by a~~. Show that it has exactly two generators. 3

(b) (i) Show that every subgroup of a cyclic group is cyclic 3

(ii) Prove that $(\mathbb{Q},+)$ is not cyclic 2

(c) Let a be an element of a group G. Then prove that

(i) $o(a)=o(a^{-1})$ 2

(ii) if $o(a)=n$ and $a^m=e$, then n is a divisor of m, e is the identity element in G. 1

(iii) if $o(a)=n$, then for a positive integer m, $o(a^m)=\frac{n}{\gcd(m,n)}$ 2

(d) (i) Find all elements of order 10 in the group $(\mathbb{Z}_{30},+)$ 3

(ii) Let G be a finite cyclic group generated by a. If $o(a)=n$, prove that $o(G)=n$ 2

(e)(i) Let H and K be two normal subgroups of a group G. Then prove that $HK=KH$ and HK is a normal subgroup G. 3

(ii) Let H be a subgroup of a Group G and $[G:H]=2$. Then show that H is normal in G. 2

(f) Let H be a subgroup of a Group G. If $x^2 \in H$ for any $x \in G$, prove that H is a normal subgroup of G and $G/H$ is commutative 5

3. Answer any three questions, taking at least one from (a) and (b) and at least one from (c) and (d).

(a) (i) State and prove Fermat's little theorem 5

(ii) (1) Let $G$ be a group and $(ab)^3 = a^3 b^3$, $\forall a, b \in G$. Show that $H = \{x^2 : x \in G\}$ is a subgroup of $G$ 3

(2) Find all subgroups of $(\mathbb{Z}, +)$ 2

(b) (i)(1) Give an example of an infinite Group in which ~~there~~ for every ~~exists a~~ subgroup, ~~whose~~ the index is finite and justify your answer. 3

(2) Let $G$ be a group and $a \in G$ be the unique element of order $n$ and $a \neq e$, show that $n = 2$ 2

(ii) State and prove Lagrange's theorem 5

(c) (i) Let $(S, *)$ be a semigroup and for any two elements $a, b \in S$, each of the equation $a * x = b$ and $y * a = b$ has a solution in $S$ for $x$ and $y$. Then prove that $(S, *)$ is a group. 5

(ii) (1) Let $G$ be a group and $a, b \in G$. Show that $(aba^{-1})^n = ab^na^{-1}$, if and only if $b = b^n$, for any integer $n$ 3

(2) Let $G$ be a group such that $a^2 = e$, $\forall a \in G$, $e$ is the identity element in $G$. Prove that $G$ is abelian. 2

(d) (i) Prove that every finite cyclic group of order $n$ is isomorphic to $(\mathbb{Z}_n, +)$ and every infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$. 5

(ii)(1) Show that $(\mathbb{R}, +)$, the group of real numbers under addition and $(\mathbb{R}^+, \cdot)$, the group of positive real numbers under multiplication are isomorphic 3

(2) Show that the group $S_3$ is not isomorphic with $(\mathbb{Z}_6, +)$. 2

(e) (i) State and prove Cayley's theorem. 5

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(ii) State and prove third ~~Isof~~ Isomorphism Theorem. 5

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Some more exercises

1. If an abelian group $G$ of order 10 contains an element of order 5, prove that $G$ must be cyclic.

Proof: Let $a \in G$ and $o(a) = 5$

Since $G$ is a finite group of even order, it contains at least one element, say $b$, of order 2 (EX 15, Page-61)

Since $o(a)$ and $o(b)$ are prime to each other and $ab = ba$, $o(ab) = 5 \times 2 = 10$ (Ex. 10, Page-57)

Since, $o(G) = 10$ and there exists an element of order 10 in $G$, $G$ must be cyclic. (Theorem ~~30B~~ 3.4, Page-20)

2. Prove that the subset $H = \{i, (12), (34), (12)(34)\}$ of $S_4$ forms a non-cyclic subgroup of $S_4$, $i$ is the identity permutation. Is the group $S_4$ cyclic?

Solution: We form the composition table for $H$ as follows:

| | $i$ | $(12)$ | $(34)$ | $(12)(34)$ |
|---|---|---|---|---|
| $i$ | $i$ | $(12)$ | $(34)$ | $(12)(34)$ |
| $(12)$ | $(12)$ | $i$ | $(12)(34)$ | $(34)$ |
| $(34)$ | $(34)$ | $(12)(34)$ | $i$ | $(12)$ |
| $(12)(34)$ | $(12)(34)$ | $(34)$ | $(12)$ | $i$ |

Here $(12)(34) = (34)(12)$

So, from the table, we see that if $a, b \in H$, then $ab \in H$

So, $H$ is a subgroup. Also, $o(12) = 2$, $o(34) = 2$ $o((12)(34)) = 2$

So, $H$ is not cyclic as there is no element of order 4.

$S_3$ is also not cyclic as it is not abelian

$[\,(124)(132) = (134) \neq (132)(124) = (243)\,]$

3. Prove that, for each positive integer $n$ there exists a cyclic group of order $n$. Proof: Let $G$ be the set of all $n$ $n$th roots of unity i.e., $G = \{x \in \mathbb{C} : x^n = 1\}$, So, $G = \{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$ where $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. $G$ forms a group with respect to multiplication. $O(G) = n$ and $o(\alpha) = n$ also. So, $G$ is cyclic. ~~So, the~~ Thus, we get the required result.

4. Prove that a non-commutative group of order $2n$, where $n$ is an odd prime, must have a subgroup of order $n$.

Proof: Let $G$ be a group of order $2n$, where $n$ is an odd prime. The divisors of $2n$ are $1, 2, n$ and $2n$. The possible orders of different elements of the group are 1, or 2, or $n$, or $2n$.
No element of $G$ can have order $2n$, because if there be an element of order $2n$, then $G$ ~~to~~ must be cyclic and hence commutative. The group contains only one element of order 1. If the order of each non-identity element be 2, then the group is commutative (Ex 4, Page 55), a contradiction.
So, there must be an element $b$ of order $n$. The cyclic subgroup $\langle b \rangle$ is a subgroup of $G$ of order $n$.

5. Prove that every proper subgroup of a group of order 6 is cyclic

Proof: Let $G$ be a group of order 6 and $H$ be a proper subgroup of $G$. By Lagrange's theorem, $O(H)$ is a divisor of 6. The divisors of 6 are 1, 2, 3 and 6. Since $H$ is a proper subgroup of $G$, $o(H) < 6$.
If $o(H) = 1$, then $H = \{e\}$, $e$ is the identity element in $G$. Hence $H$ is cyclic
If $O(H) = 2$, then $H$ is a group of prime order and so $H$ is cyclic.
If $O(H) = 3$, then also $H$ is a group of prime order and so $H$ is cyclic.
Thus in any case, $H$ is cyclic.
Note: Every proper subgroup of $S_3$ is cyclic.