SOLVE IT → Model Question Paper for CC4 — (2)

Full marks : 65

1. Choose the correct alternative. Justify your answer 2×10

(a) The system which is a group, is

(i) $(\mathbb{Z}, \circ)$ where $a \circ b = a + b + ab$, $a, b \in \mathbb{Z}$

(ii) $(\mathbb{R}^*, \circ)$ where $a \circ b = |ab|$, $a, b \in \mathbb{R}^* = \mathbb{R} - \{0\}$

(iii) $(\mathbb{Z}, \circ)$ where $a \circ b = a + b + 1$, $a, b \in \mathbb{Z}$

(iv) $(\mathbb{R}, \circ)$ where $a \circ b = 2(a+b)$, $a, b \in \mathbb{R}$

(b) Let $(G, \circ)$ be a group and $c \in G$. Define a binary operation $*$ on G by $a * b = a \circ c \circ b$, $a, b \in G$. Then

(i) $(G, *)$ is a group and c is the identity element.

(ii) $(G, *)$ is not a group

(iii) $(G, *)$ is a group and c^2 is the identity element

(iv) $(G, *)$ is a group and c^{-1} is the identity

(c) An element a in a group $(G, \circ)$ is said to be an idempotent element if $a \circ a = a$. The number of idempotent elements in a group is

(i) 2 (ii) 0 (iii) 1 (iv) 3

(d) Let $(G, \circ)$ be a group. Define a mapping $\lambda_a : G \to G$ by $\lambda_a(x) = a^{-1} \circ x \circ a$ for a fixed element $a \in G$. Then

(i) λ_a injective but not surjective

(ii) λ_a is bijective

(iii) λ_a is neither injective nor surjective

(iv) λ_a is surjective but not injective.

(e) The number of elements of order 5 in the group $(\mathbb{Z}_{30}, +)$ is

(i) 2 (ii) 5 (iii) 4 (iv) 6

(f) If b be an element of a group and $o(b) = 20$, then $o(b^{15})$ is

(i) 4 (ii) 6 (iii) 5 (iv) 8

(g) The example of a non-cyclic abelian group is

(i) S_3

(ii) D_4

(iii) $(\mathbb{Z}_4, +)$

(iv) Klein's 4-group

(h) Index $[G:H]$ for the group $G = S_3$ and $H = \langle \rho_1 \rangle$ is

(i) 1 (ii) 2 (iii) 4 (iv) none of (i), (ii) and (iii)

(i) The solution of the equation $\rho_3 x = \rho_1$ in S_3 is

(i) ρ_1 (ii) ρ_4 (iii) ρ_1 (iv) ρ_3

(j) which of the following statements is true in S_3?

(i) S_3 has a non-trivial centre

(ii) All subgroups of S_3 is normal

(iii) S_3 has a trivial centre

(iv) none of the above

2. Answer one question from (a) and (b), one question from (c) and (d); and one question from (e) and (f)

(a) (i) Let G be a finite semigroup in which both the cancellation laws hold. Then show that G is a group. 3

(ii) Let G be a group and $a \in G$. Define a mapping $\rho_a : G \to G$ by $\rho_a(x) = xa$, $x \in G$. Prove that ρ_a is a bijection. 2

(b) (i) Let a be an element of a group G. Then for integers m and n, prove that $a^m a^n = a^{m+n}$ 3

(ii) Find the orders of each element of $(\mathbb{Z}_6, +)$ 2

(c) (i) Let H and K be two subgroups of a group G. Then prove that HK is a subgroup of G if and only if $HK = KH$ 4

(ii) Give examples of two subgroups H and K of a group G such that HK and KH both are not subgroups of G. 1

(d) Let G be a finite cyclic group of order $n > 1$, generated by a. Then prove that, for a positive integer r, a^r is also a generator if and only if r is less than n and prime to n. 5

(e) If H be a subgroup of a cyclic group then show that G/H exists and it is cyclic. Show that the converse is not true. 5

(f) Let G and G′ be two groups and $\phi: G \to G'$ be ~~an~~ a ~~onto~~ homomorphism then show that Kerϕ is a normal subgroup of G. If ϕ is also onto, then show that if H is a normal subgroup of G then $\phi(H)$ is also a normal subgroup of G′. 5

3. Answer any three questions taking at least one from (a) and (b) and at least one from (c) and (d)

(a) 1.(i) In ~~a~~ a group G, a and b are distinct elements of order 2. If a and b commute, prove that $o(ab) = 2$. 2
If a and b do not commute, prove that $o(aba^{-1}) = 2$

(ii) Let G be a group and $a, b \in G$. If $o(a) = 3$ and $aba^{-1} = b^2$, find $o(b)$ if $b \neq e$, e is the identity element in G. 3

2. Show that a finite group G of order n is cyclic if and only if $\exists$ an element ~~of~~ b ~~of~~ in G such that $o(b) = n$. 5
Hence prove that Klein's 4-group and the group D_4 is not cyclic.

(b). 1. Let H be a subgroup of a Group G and $a, b \in G$.
(i) ~~Then~~ prove that $b \in aH$ if ~~and~~ and only if $a^{-1}b \in H$
(ii) Show that the relation ρ defined on G by $a\rho b$ if and only if $a^{-1}b \in H$, for $a, b \in G$, is an equivalence relation on G, and each equivalence class is a left coset of H. 5

2. (i) Show that every group of prime order is cyclic. But show that the converse is not true. 3

(ii) P and Q are two subgroups of a group G and $o(P)$ and $o(Q)$ are relatively prime. Prove that $P \cap Q = \{e\}$, e is the identity element in G. 2

c) 1. (i) Let H be a cyclic subgroup of a group G. If H be normal in G prove that every subgroup of H is a normal in G. 3

(ii) Prove that $SL(n, \mathbb{R})$ is a normal subgroup of $GL(n, \mathbb{R})$ where $GL(n, \mathbb{R})$ is the group of all real non-singular $n \times n$ matrices with respect to multiplication and $SL(n, \mathbb{R})$ is the group of all real non-singular matrices A, with $\det A = 1$, with respect to multiplication. 2

2. (i) Let G be a group and H be a normal subgroup of G. If L be a subgroup of G such that $H \subset L \subset G$ then prove that L/H is a subgroup of G/H. 2

(ii) Let G be a cyclic group of order 12 generated by a and H be the cyclic subgroup of G generated by a^4. Prove that H is normal in G. Write out all cosets of H in G. Verify that the quotient group G/H is a cyclic group of order 4. 3

(d) 1 (i) Let H be a normal subgroup of a group G. Define the mapping $\theta : G \to G/H$ by $\theta(x) = xH$, $x \in G$. Show that θ is an onto homomorphism with $\mathrm{Ker}\,\theta = H$. 2

(ii) Let $\phi : G \to G'$ be a homomorphism of group G onto a group G' and $H = \mathrm{Ker}\,\phi$. Let θ be homomorphism of G onto G' defined by $\theta(x) = xH$, $x \in G$. Show that $\exists$ an isomorphism $\psi : G/H \to G'$ such that $\phi = \psi\theta$. 3

2. State and prove First Isomorphism Theorem 5

(e) 1. (i) Let G be a group and the mapping $\alpha : G \to G$ is defined by $\alpha(x) = x^{-1}$, $x \in G$. Prove that α is an automorphism if and only if G is abelian 3

(ii) State and prove Second Isomorphism Theorem 2

2. (i) If G be an infinite cyclic group, prove that $Aut(G)$ is a group of order 2 3

(ii) Show that the groups $(\mathbb{Z}_4, +)$ and Klien's 4-group V are not isomorphic. 2

Some more problems

1. Prove that there does not exist ~~and on~~ an onto homomorphism from the group S_3 to the group $(\mathbb{Z}_6, +)$

Let $G = S_3$ and $G' = (\mathbb{Z}_6, +)$

If possible let $\phi : G \to G'$ be an onto homorphism,

$\bar{1} \in \mathbb{Z}_6$ and $o(\bar{1}) = 6$. As ϕ is onto,

let $x \in S_3$ such that $\phi(x) = \bar{1}$

As $x \in S_3$, $O(x)$ is either 1 or 2 or 3.

Since ϕ is a homomorphism, $o(\phi(x))$ is a divisor of $o(x)$.

This can not happen as $o(\phi(x)) = 6$. So, there does not any onto homorphism between S_3 and $(\mathbb{Z}_6, +)$

2. Let G be a group and H be a non-empty subset of G. A relation ρ defined on G by $a\rho b$ if and only if $ab^{-1} \in H$ for $a, b \in G$, is an equivalence relation. Prove that H is a subgroup of G.

~~Hence~~ Proof: Here $H \neq \phi$ (given). Let $c \in H$. As ρ is reflexive,

$c\rho c$ holds $\Rightarrow cc^{-1} = e \in H$, e is the identity in G.

Let $a, b \in H$. Case 1 Let $a = b = e$, then $ab^{-1} = ee^{-1} = e \in H$

Case 2 Let $a = e$, $b \neq e$ Then $ab^{-1} = b^{-1}$ Now $be^{-1} = b \in H$, so $b\rho e$. So, $e\rho b$ (ρ is symmetric). So, $eb^{-1} \in H$ or $b^{-1} \in H$. So, $ab^{-1} \in H$

Case 3 Let $a \neq e$, $b = e$. The proof is similar as Case 2.

Case 4. Let $a \neq e$, $b \neq e$. As $ae^{-1} = a \in H$ and $be^{-1} = b \in H$, so,

$a\rho e$, and $b\rho e$. So $a\rho e$ and $e\rho b$ (ρ is symmetric)

So, $a\rho b$ (ρ is transitive). Hence $ab^{-1} \in H$. So H is a subgroup of G.