Some MCQ type Questions

Choose the correct alternative. Justify your answer   2 marks

1. Let G be any set having at least two elements. Define the binary operation 'o' by $a o b = b$, $a, b \in G$. (G, o) is not a group because

(i) 'o' is not associative

(ii) The equation $a o x = b$ does not have a solution in G, $a, b \in G$.

(iii) Cancellation ~~laws~~ (G, o) does not satisfy cancellation laws.

(iv) None of the above.

2. Number of idempotent elements in a group of order 32 is

(i) 8

(ii) 1

(iii) 6

(iv) 12

3. For the group $G = \{x \in \mathbb{C} : x^8 = 1\}$ with respect to complex multiplication, the inverse of $\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$ is

(i) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

(ii) $\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

(iii) $\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

(iv) $\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

4. The solution of the equation $(1234)x = (12)(34)$ in S_4 is

(i) (14) (ii) (24) (iii) $(2,3)$ (iv) (13)

5. The number of elements of order 2 in Klein's 4-group is

(i) 3 (ii) 2 (iii) 1 (iv) 0

6. The number of elements of order 6 in S_3 is

(i) 1 (ii) 2 (iii) 3 (iv) 0

7. An example of a nonabelian non-cyclic group is

(i) $(\mathbb{Q}, +)$ (ii) $(\mathbb{R}, +)$ (iii) S_3 (iv) None of the above.

8. The inverse of $\rho_2 = (132)$ in S_3 is

(i) (132) (ii) (123) (iii) (13) (iv) (12)

9. The inverse of $\alpha = (1342)(5786)$ in S_8 is

(i) $(1324)(5687)$ (ii) $(1234)(5786)$ (iii) $(1243)(5687)$ (iv) $(1432)(5867)$

10. In a group G, a is an element of order 30. The order of a^{24} is

(i) 15 (ii) 5 (iii) 6 (iv) none of the above

11. ~~If~~ The number of elements of order 6 in $(\mathbb{Z}_{30}, +)$ is

(i) 4 (ii) 3 (iii) 2 (iv) none of the above

12. The centre of the group $(\mathbb{R}, +)$ is

(i) $(\mathbb{Z}, +)$ (ii) $(2\mathbb{Z}, +)$ (iii) $(\mathbb{Q}, +)$ (iv) None of the above

13. If ~~a,~~ a and b be two distinct elements of order 2 in a commutative group G. Then order of the element ab is

(i) 1 (ii) 4 (iii) 2 (iv) None of the above

14. order of the element 2 in $(\mathbb{Z}, +)$ is

(i) 1003 (ii) 2001 (iii) 3 (iv) None of the above

15. The two generators of the infinite cyclic group $(6\mathbb{Z}, +)$ is

(i) 2, −2 (ii) 3, −3 (iii) 6, −6 (iv) None of the above

16. The number of generators of the cyclic group $(\mathbb{Z}_{36}, +)$ is

(i) 12 (ii) 11 (iii) 10 (iv) None of the above

17. Which of the following statements is not true?

(i) S_3 is not cyclic

(ii) $(5\mathbb{Z}, +)$ is cyclic

(iii) Klein's 4-group is not cyclic

(iv) D_4 is cyclic

18. A cyclic group G has only one generator. Then

(i) $o(G) = 1$ or $o(G) = 2$ (ii) $o(G) = 2$ or $o(G) = 3$ (iii) $o(G) = 3$ or $o(G) = 4$

(iv) None of the above.

19. The group in which there exists at least one element of finite order other than the identity element, is
(i) $(\mathbb{Z},+)$ (ii) $(2\mathbb{Z},+)$ (iii) $(3\mathbb{Z},+)$ (iv) None of the above

20. Order of the Quotient group S_{58}/A_{58} is
(i) 3 (ii) 5 (iii) 2 (iv) 4

21. The centre $Z(S_3)$ of the group S_3 is
(i) $\{\rho_0, \rho_3\}$ (ii) $\{\rho_0, \rho_4\}$ (iii) $\{\rho_0, \rho_5\}$ (iv) None of the above

22. The Quotient group $\mathbb{Z}/2\mathbb{Z}$ is
(i) non-abelian (ii) cyclic (iii) non-cyclic (iv) None of the above

23. The order of the Quotient group $S_3/Z(S_3)$ is
(i) 2 (ii) 3 (iii) 6 (iv) None of the above

Some problems of 5 marks

1.(i) Prove that if G is an abelian group, then $(ab)^n = a^n b^n$ for any $a, b \in G$ and for any integer n. 3.

(ii) Prove that the set D of all odd integers forms a commutative group with respect to $*$ defined by $a*b = a+b-1$ for $a, b \in D$

2.(i) Prove that the set of all rational numbers, other than 1, forms a commutative group with respect to the composition $*$ defined by $a*b = a+b-ab$, $a, b \in \mathbb{Q}-\{1\}$ 3

(ii) Prove that a group is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ $\forall a, b \in G$. 2

3.(i) Give an example of a group G in which $o(a)$ and $o(b)$ is finite but $o(ab)$ is infinite for some $a, b \in G$ 3

(ii) Give an example of a group G in which $o(a).o(b) \neq o(ab)$ for some $a, b \in G$ 2

3. (i) Let G be a group and $a, b \in G$. If $a^{-1}b^2a = b^3$ and $b^{-1}a^2b = a^3$, show that $a = b = e$, e is the identity in G. 3

Solution: $a^{-1}b^2a = b^3 \Rightarrow (a^{-1}b^2a)^4 = (b^3)^4 \Rightarrow a^{-1}b^8a = b^{12}$

$\Rightarrow a^{-3}b^8a^3 = a^{-2}b^{12}a^2 \Rightarrow a^{-2}(a^{-1}b^8a)a^2 = a^{-2}b^{12}a^2 = a^{-1}(a^{-1}b^{12}a)a$

$\Rightarrow a^{-3}b^8a^3 = a^{-1}(a^{-1}b^2a)^6a \Rightarrow a^{-3}b^8a^3 = a^{-1}b^{18}a = (a^{-1}b^2a)^9 = b^{27}$

Again $a^{-3}b^8a^3 = (b^{-1}a^2b)^{-1}b^8(b^{-1}a^2b) = (b^{-1}a^{-2}b)b^8(b^{-1}a^2b)$

$= b^{-1}a^{-2}b^8a^2b = b^{-1}a^{-1}(a^{-1}b^2a)^4ab = b^{-1}(a^{-1}b^{12}a)b$

$= b^{-1}(a^{-1}b^2a)^6b = b^{-1}b^{18}b = b^{18}$

So, $b^{27} = b^{18} \Rightarrow b^9 = e$

$\Rightarrow (b^3)^3 = e \Rightarrow (a^{-1}b^2a)^3 = e \Rightarrow a^{-1}b^6a = e$

$\Rightarrow b^6 = e$ So, $b^9 = b^6 \Rightarrow b^3 = e$

or, $a^{-1}b^2a = e \Rightarrow b^2 = e$

So, $b^3 = b^2 \Rightarrow b = e$

As we replace a by b in the conditions, they remain the same, so, similarly we get $a = e$.

(ii) If b be an element of a group and $o(b) = 20$, find $o(b^8)$ 2

Solution, $o(b^8) = \dfrac{o(b)}{\gcd(o(b), 8)} = \dfrac{20}{\gcd(20, 8)} = \dfrac{20}{4} = 5$

4. (i) Let G be an abelian group. Prove that the subset $H = \{x \in G : x = x^{-1}\}$ forms a subgroup of G. 2

(ii) Let G be a group and H be a subgroup of G. Let a be some fixed element of G. Prove that $aHa^{-1} = \{aha^{-1} : h \in H\}$ is a subgroup of G. If H be a finite subgroup of G, prove that $o(gHg^{-1}) = o(H)$. 3

5. (i) Let G be a group in which $(ab)^3 = a^3b^3$, for all $a, b \in G$. Show that $H = \{x^6 : x \in G\}$ is a subgroup of G. 2

(ii) Let a, b be fixed positive integers and $H = \{ax + by : x, y \in \mathbb{Z}\}$. Show that $(H, +)$ is a subgroup of ~~o~~ the group $(\mathbb{Z}, +)$. Show that if $\gcd(a, b) = 1$, then $H = \mathbb{Z}$ 3

6. (i) Describe the left cosets and the right cosets of H in G and find $[G : H]$ where $G = S_3$ and $H = \langle \rho_1 \rangle$ 3

(ii) Find all subgroups of the group $(\mathbb{Z}_{10}, +)$. 2

7 (i) H and K are different subgroups of a group G such that $o(H) = o(K) = p$ where p is a prime. Show that $H \cap K = \{e\}$. Deduce that if G has exactly m distinct subgroups of prime order p then the total number of elements of order p is $m(p-1)$. 3

(ii) Let G be an infinite cyclic group $\langle a \rangle$ and let H be the subgroup $\langle a^m \rangle$, where m is a positive integer > 1. Prove that $H, aH, a^2H, \ldots, a^{m-1}H$ is a complete list of distinct ~~left~~ left cosets of H in G. 2.

8. (i) Prove that a group of order 27 must have a group of order 3. 3

Solution: Let G be the group of order 27 and let $a \in G$ and $a \neq e$, e is the identity element. As $o(a)$ divides 27 and $a \neq e$, either $o(a) = 3$ or $3^2 = 9$ or $3^3 = 27$. If $o(a) = 3$ then $\langle a \rangle$ is the required group. If $o(a) = 9$ then $o(a^3) = \dfrac{9}{\gcd(3,9)} = \dfrac{9}{3} = 3$

So, $\langle a^3 \rangle$ is the required group. If $o(a) = 27$ then $o(a^9) = \dfrac{27}{\gcd(9,27)} = 3$. So $\langle a^9 \rangle$ is the required group.

(ii) Prove that $\{\rho_0, \rho_4\}$ is not a normal subgroup of S_3 2