Some More Problems

1. Prove that the intersection of a finite number of subgroups of finite index is a subgroup of finite index. Is the intersection of an infinite number of subgroup of finite index necessarily also of finite index?

Hint: Let H & K be two subgroups of finite index. Now $x(H \cap K) = xH \cap xK$. This implies $H \cap K$ is a subgroup of finite index and then use induction for finite number of subgroups.

The result is not true for an infinite number of subgroups each of finite index. To see this, consider the group $(\mathbb{Z}, +)$. The subgroup $n\mathbb{Z}$ has index n but $\bigcap_{n=1}^{\infty} n\mathbb{Z} = \{0\}$ which is not of finite index in $\mathbb{Z}$.

2. Let G be a group and H be a subgroup. Prove that the only left coset of H in G is a subgroup of G is H itself. Define a mapping ϕ from the set of left cosets of H in G to the set of right cosets of H in G by $\phi(xH) = Hx^{-1}$. Show that ϕ is well defined and it is bijective. Is $\psi(xH) = Hx$ a well defined mapping from the left cosets of H in G to the right cosets of H in G?

Solution: Let xH be a subgroup of G. Then $e \in xH$, e is the identity element in G. So, $x^{-1} \in x^{-1}xH = H$ and hence $x \in H$ (H is a subgroup). So $xH = H$.

Let $xH = yH \Rightarrow x^{-1}y \in H \Rightarrow Hx^{-1}y = H \Rightarrow Hx^{-1} = Hy^{-1} \Rightarrow \phi(x) = \phi(y)$

So ϕ is well defined. Let $\phi(x) = \phi(y) \Rightarrow Hx^{-1} = Hy^{-1}$

$\Rightarrow x^{-1}(y^{-1})^{-1} \in H \Rightarrow x^{-1}y \in H \Rightarrow xH = yH$. So ϕ is injective.

For Hx $\quad \phi(x^{-1}H) = H(x^{-1})^{-1} = Hx$. So, ϕ is surjective.

So, ϕ is bijective.

ψ may not be a mapping. Consider the example $G = GL(2, \mathbb{Q})$ and $H = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{Q}, ac \neq 0 \right\}$

We have $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} H = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} H$

as $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{t} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in H$

But $H \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq H \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

as. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{t} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \notin H$

3. Find a group G with subgroups H and K such that HK is not a subgroup.

Solution: Consider the subgroup $H = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{Q} \right\}$ and $K = \left\{ \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} : a \in \mathbb{Q} \right\}$ Check that $HK = \left\{ \begin{bmatrix} 1+ab & b \\ a & 1 \end{bmatrix} : a, b \in \mathbb{Q} \right\}$

But HK is not a subgroup of G, since, for example if we take A and B as

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Here $A, B \in HK$

but $AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \notin HK$

4. Consider the subgroup $H = \langle \rho_5 \rangle$ of S_3. Show how the left cosets of H partition S_3. Show also how the right cosets of H partition S_3. Deduce that H is not a normal subgroup of S_3.

Solution: The left cosets are $\rho_0 H = \{\rho_0, \rho_5\} = \rho_5 H = H$

$$\rho_4 H = \{\rho_4, \rho_1\} = \rho_1 H$$

$$\rho_3 H = \{\rho_3, \rho_2\} = \rho_2 H$$

So, $S_3 = H \cup \rho_4 H \cup \rho_3 H$ and $H, \rho_4 H, \rho_3 H$ are pairwise disjoint

Again, the right cosets are $H\rho_0 = H\rho_5 = \{\rho_0, \rho_5\} = H$

$$H\rho_4 = \{\rho_4, \rho_2\} = H\rho_2$$

$$H\rho_1 = \{\rho_3, \rho_1\} = H\rho_3$$

So, $S_3 = H \cup H\rho_4 \cup H\rho_1$ and $H, H\rho_4, H\rho_1$ are pairwise disjoint.

Here the left coset $\rho_3 H$ is not a right coset

So, H is not normal in S_3

5. Let G be a group and let H be a subgroup of G. If $g \in G$ is such that $o(\langle g \rangle) = n$ and $g^m \in H$ where m is an integer relatively prime to n, show that $g \in H$.

Proof: Since m and n are prime to each other, $\exists$ integers a and b such that $am + bn = 1$. Now $g^n = e$, e is the identity element in G. We have,

$$g = g^1 = g^{am+bn} = g^{am} g^{bn} = (g^m)^a (g^n)^b = (g^m)^a$$

Since it is given that $g^m \in H$, we have $(g^m)^a \in H$ and hence $g \in H$

6. Let G be a group. Prove that
(i) If H is a subgroup of G then $HH = H$
(ii) If X is a finite subset of G with $XX = X$, then X is a subgroup of G.
Show that (ii) fails for infinite subsets of X.

Solution: Given $x \in X$, we have $xX \subseteq XX = X$. Now the mapping $f: X \to xX$ defined by $f(y) = xy$, $y \in X$ is injective as $f(y_1) = f(y_2) \Rightarrow xy_1 = xy_2 \Rightarrow y_1 = y_2$ (by cancellation law).
So, we get $o(xX) = o(X)$ and hence, since X is finite, $xX = X$. Consequently $x = xa$ for some $a \in X$. The cancellation law gives $a = e$ and so, we have $e \in X$, e is the identity element in G. We now observe that $e \in xX$ so, $xy = e$ for some $y \in X$, which gives $y = x^{-1} \in X$
It now follows from the fact that $XX \subseteq X$, X is a subgroup of G.
That (ii) no longer holds when X is infinite may be seen by taking the group $(\mathbb{Z}, +)$ and taking X to be the set of non-negative integers.

7. Let G be a finite group with more than one element. Show that G has an element of prime order.

Solution: Let $a \in G$, and $a \neq e$, e is the identity element in G.
Consider $a, a^2, a^3, \ldots$. Since G is finite, $\exists$ integers i and j, $i > j$ such that $a^i = a^j$. So, $a^{i-j} = e$
Let $i - j = n$, then n is an integer and $n > 0$ such that $a^n = e$. Since $a \neq e$, $n > 1$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where p_i's are distinct primes

So, $a^{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}} = e \Rightarrow \left(a^{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r - 1}}\right)^{p_r} = e$

$\Rightarrow o\left(a^{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r - 1}}\right) = 1$ or p_r

If $o\left(a^{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r - 1}}\right) = p_r$, then the result follows.

If $o\left(a^{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r - 1}}\right) = 1 \Rightarrow a^{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r - 1}} = e$, then

proceeding as above, we get an element of prime order as $a \neq e$.

8. Suppose that G is a finite group with the property that every non-identity element has prime order. If $Z(G)$ is non-trivial, prove that every non-identity element of G has the same order.

Solution : As $Z(G)$ is non-trivial, let $a \in Z(G)$ and $a \neq e$

Let $o(a) = p$ (p is a prime). Let $b \in G$ and $o(b) = q$ (q is a prime). Since $a \in Z(G) \Rightarrow ab = ba$

So, $(ab)^{pq} = a^{pq} b^{pq} = e$

$\Rightarrow o(ab)$ divides pq

$\Rightarrow o(ab) = 1, p$ or q (as every element has prime order, so, $o(ab) \neq pq$)

If $o(ab) = 1$ then $a = b^{-1} \Rightarrow o(a) = o(b^{-1}) = o(b) \Rightarrow p = q$

If $o(ab) = p$ then $(ab)^p = e \Rightarrow a^p b^p = e \Rightarrow b^p = e$

$\Rightarrow q$ divides $p \Rightarrow q = p$ (q, p are primes)

Similarly, if $o(ab) = q$, $p = q$. So, every non-identity element of G has the same order.